Solutions Manual and Reference for The Economics of Business Valuation

Edited by Erin A. Grover

With Contributions from Patrick L. Anderson, Samantha Superstine, and Jeff Johnson

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Anderson Economic Group LLC
1555 Watertower Place
East Lansing, Michigan 48823
Tel: (517) 333-6984
Fax: (517) 333-7058


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Preface

A Brief Introduction

SECTION 1.1. Purpose

The idea to create a companion volume was initially formulated due to the original length of *The Economics of Business Valuation* (EBV). Additionally, given the novelty of the value functional method, the author of EBV, Patrick L. Anderson, wanted to provide readers with additional examples. Therefore, the purpose of this compilation of work is to provide extensive material and references for scholars, readers, and practitioners who purchase EBV.

This companion volume may also be useful to those interested in notions of value in various fields, use of the net present value (NPV) rule, and/or the mathematics of dynamic programming.

SECTION 1.2. Design and Content

The design of this companion volume attempts to mirror the organization of EBV. As you recall, EBV arranges every three chapters into a “part,” which illustrates a shared theme throughout those chapters. In this pub-

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lication, each chapter contains supplemental or related material to the corresponding part in EBV; Chapter 1 corresponds with EBV Part I.

For example, Chapter 2 of the Solutions Manual discusses the principal-agent problem in firms, as well as presents unique data on U.S. businesses. This chapter is meant to supplement EBV’s Part II “The Nature of the Firm.” To further illustrate this design, we named each chapter after its counterpart in EBV.

As described in Appendix B “Guide to the Solutions Manual,” this compilation of work covers a wide range of fields. The extended material provided here includes a discussion of the following topics:

- common notions of value (Section 1.1. on page 1)
- the “winner’s curse” and the human condition (Section 1.2. on page 10)
- principal-agent and moral hazard problems (Section 2.1. on page 15)
- business entity classifications and other forms of organization for private businesses (Section 2.3. on page 22)
- types of U.S. firms in terms of size, organization, employment, and survival rates (Section 2.2. on page 17, Section 2.4. on page 28, and Section 2.5. on page 32)
- neoclassical economics in the real world (Section 3.1. on page 42)
- the math behind the neoclassical model in economics (Section 3.2. on page 43)
- complete market pricing formulas (Section 4.1. on page 51)
- applications of Mathematics to problems in Finance (Chapter 5)
- existing valuation standards (Section 5.7. on page 79)
- software for business valuation (Appendix B)

In addition to these topics, we provide several other value functional examples and explanations that expand upon Part VI “Applications” of EBV.\(^2\) We include a set of questions and problems for each EBV chapter, as well as hints for several of these problems in Chapter 7. We also provide a guide to the mathematics of value, which expands upon the summary of notation and important formulas presented in Appendix A “Key Formula and Notation Summary” in EBV.\(^3\) Lastly, we include a

\(^2\) See Chapter 6, “Applications: Intermediate Results of Value Functional Examples in EBV.”

\(^3\) See Appendix A “Guide to Mathematics of Value.”
section for errata and additional references, should any errors be found or clarifications needed in EBV or this companion volume.4

SECTION 1.3. Acknowledgements

I must acknowledge that this was most definitely a group effort and the following individuals greatly contributed to this companion volume:

Samantha Superstine, who compiled and analyzed the business data for Chapter 2 “The Nature of the Firm,” as well as authored the bulk of it. She also graciously offered her time to assist in proofreading and editing several of the other chapters.

Jeffrey Johnson, who authored Chapter 6 “Applications.” He ran the analysis for each value functional example, as well as provided a description for each that was understandable to even the non-programmer, which was so greatly appreciated.

Manav Garg, who graciously reviewed the math and equation heavy content of this companion volume. He checked for accuracy and provided comments for Chapters 4 and 5, as well as Appendix A.

Christi Lilleboe, Tyler Theile, and Kimberly Kvorka each helped in a multitude of tasks. Christi was essential to streamlining the format of this companion volume. She also assisted in proofreading and editing text as well as reformatting equations using MathType. Tyler helped to ensure that this companion volume made it on the web, as well as had an ISBN. Kim assisted in organizing original writings and a multitude of other administrative tasks.

Last, but certainly not least, Patrick L. Anderson, the author of The Economics of Business Valuation, without which this companion volume would not have made much sense. He was kind and encouraging to allow me the opportunity to create this publication. As part of that initiative, he provided us with the bulk of writing that constitute Chapters 1, 3, 4, 5, and 7, in addition to Appendices A and B.

4 See Appendix C “Errata.”
Chapter 1

Theories of Value

by Patrick L. Anderson

An Introduction

There are many concepts of value, just as there are many theories of value that are valid within the confines of their defined assumptions and limitations. In this chapter, we first discuss common notions of value, and how value is determined in various fields. This gives historical and practical context to perceptions of value. We then describe how different people could attribute different values to the same item, and particularly how that differentiation applies in the case of business valuation.

Section 1.1. Common Notions of Value

a. What is a “standard” of value?

Given the amorphous nature of the simple term “value,” it should not be surprising that several different specific standards of “value” are in common use. By “standard of value,” we mean the practical definition of value, and not the method of determining that value, nor the theoretical basis for that practical definition.1 In this section, we discuss standards of value that are commonly used by economists, businessmen and women, investors, and tax authorities.
b. Market Value in Economics

The most critical aspect of the market, and the market price, is the existence of competing buyers and sellers. It is this competition among self-interested parties that makes the sales price so efficient. Indeed, much of the triumph of capitalism around the world over the last two centuries can be traced to the efficiency of markets, and the mechanism of market prices.

The intellectual basis for free markets as the economic system that would most enrich society was laid in the seminal work of Scottish moral philosopher Adam Smith, who in 1776 penned the magisterial Wealth of Nations. In particular, Smith identified free markets as the “invisible hand” that allocates world resources in a manner both equitable and encouraging of prosperity.2

The Primacy of the Market

Market value is the standard on which we will base the majority of the analysis in this publication, and is a deeply established principle in economics. In general, the “market value” of any asset is the price at which it exchanges among willing parties in a market; a market, in turn, consists of multiple potential buyers and sellers, and facilitates reasonably good information about the asset itself. The legal definition of “fair market value” used in the United States closely matches this economic concept, and will be described further below.

The “Free Market” and Institutions

Note that the existence of a market does not guarantee that an asset is actually sold. Indeed, if a seller is compelled to sell, it is no longer a true market. The lack of compulsion means that buyers and sellers can pursue

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1 This concept, unfortunately, does not have a consistent name or definition among professional valuation standards. The Appraisal Foundation’s USPAP (2010) uses the term “type or definition of value.” “Standard of value” is defined as “identification of the type of value being used” in the ASA Business Valuation Standards (2009) glossary. The IVSC International Valuation Standards (2010 exposure draft) uses the term “basis of value” and defines it as “statement of the fundamental measurement assumptions of a valuation.”

2 We discuss Smith and his contemporaries in EBV Chapter 7 “Value in Classical Economics.” The findings of welfare economics are discussed in EBV Chapter 8 “Value in Neoclassical Economics.”
their own interests in a true market, and justifies the use of the terms “free market” and “free market price.”

Furthermore, a market does not require perfect information, a lack of transaction costs, government-sanctioned standards for the product, or most other attributes that exist in the markets for many consumer products. A market does require, however, certain institutional factors, including property rights and the enforcement of contracts—two critical features of society that are normally secured by government. We discuss this topic in EBV Chapter 4 “The Nature of the Firm.”

c. The “Fair Market Value” Legal Standard

The “fair market value” concept from economics is one of the most powerful and useful to emerge from that field. Indeed, it is now codified in countless laws, standards, court cases, individual contracts, and scholarly treaties in the United States and in other countries.

This legal standard is essentially identical with the economic concept: fair market value means the value in a market with willing buyers and willing sellers, each with sufficient information, and neither under any compulsion. This standard has been refined in countless court cases, IRS rulings, and reference books.

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3 The intellectual basis for free markets as the economic system that would most enrich society was laid in the seminal work of Scottish moral philosopher Adam Smith, who in 1776 penned the magisterial *Wealth of Nations*. In particular, Smith identified free markets as the “invisible hand” that allocates world resources in a manner both equitable and encouraging of prosperity.

4 The seminal reference for this definition is the IRS Revenue Ruling 59-60 (1959) 59-1 CB 237, which defines fair market value as:
   
   The price at which the property would change hands between a willing buyer and a willing seller when the former is not under any compulsion to buy and the latter is not under any compulsion to sell, both parties having reasonable knowledge of relevant facts.

   Court decisions frequently state in addition that the hypothetical buyer and seller are assumed to be able, as well as willing, to trade and to be well informed about the property and concerning the market for such property.”

5 For example, Treasury Regulation 20.2031-1 defines fair market value as:
   
   “The price at which the property would change hands between a willing buyer and a willing seller, neither being under any compulsion to buy or to sell and both having reasonable knowledge of relevant facts.”
We note here several specific observations about this standard:

- **“Fair market value”** must be set in a market, meaning *multiple potential buyers and sellers*. Multiple potential buyers, however, does not mean multiple offers to buy, rather that multiple persons are interested in purchasing similar assets and are capable of purchasing this one.

- Another implication of “market” is that the buyers and sellers are independent of each other. This is sometimes included explicitly in the definition of an “arm’s length” transaction. Note that “arm’s length” does not mean the parties do not know each other, have no other mutual business interests, or the like, but rather that decisions of whether to buy or sell are made with individual interests in mind.

- The market must include willing parties to complete the transaction. Transactions by non-willing sellers—such as those compelled by eminent domain decrees, or by financial distress—do not constitute a market.

- In addition, the “sufficient information” phrase requires that market participants have adequate information, though not *perfect* information. Several “sales” of the Brooklyn Bridge, for example, would not demonstrate a market price if all the sales were to gullible “buyers” of a worthless piece of paper.6

The market value of any specific asset is, in general, different than the value of that asset to an individual owner. Individuals often place peculiar emphasis on certain assets, and therefore those assets may be “worth” more to them than their respective market values. (See the discussion of *investment value* or *intrinsic value*, below.)

d. **“Investment Value”**

The term “investment value” has an important economic implication, which we will note in discussing the value of privately-held firms and

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6 References to “selling the Brooklyn Bridge” have appeared in American literature for over a century. A few recent efforts to track the original source arrived at contradictory results. Some authors attribute the story to the exploits of William McCloudy and George C. Parker, who were said to have sold the bridge to recent immigrants (Parker giving the buyer a bill of sale saying “one bridge in good condition”) around 1900. According to the story, McCloudy was subsequently convicted of larceny in 1901. Others say even this story is a myth, perhaps even a braggadocio among professional con men.

A skeptical but entertaining review of these stories was penned by novelist Gabriel Cohen (2005); less skeptical accounts are from the language scholar Barry Popik (2004) and the urban-myth-investigator web site “Straight Dope” (2007).
those without widely-traded securities. It also has related technical meanings in Finance and Accounting.

**Economic Concept**

In general, the term “investment value” refers to the value of an asset as an investment, distinguishing it from the value in use for personal consumption. This distinction is important when deriving a theory of investment behavior. As discussed in EBV Chapter 8 “Value in Neoclassical Economics,” the purpose of investment is to secure future consumption, and not to achieve other goals.

The fallacy of ignoring this principle becomes clear when one attempts to use standard finance techniques to explain the behavior of investors. Explaining investor behavior requires taking this principle into account when regarding assets and other considerations they consider to be important, such as a house, reputation, standard of living, loyalty, or the use of time. In such matters, a person may quite rationally ignore a “rule” of finance (such as the NPV rule) when it conflicts with personal interests. However, the very same person may attempt to follow the very same “rule” when dealing with an investment portfolio. Contemporary authors in finance seem more prone to erroneously missing this distinction than those of many decades ago; for example, both Irving Fisher and John Burr Williams made such a distinction in their writings. 7

**Investment Value: International Valuation Standard**

The term “investment value” also has a technical definition that is related to the economic concept described above. The following is excerpted from the International Valuation Standards:

> Investment Value, or Worth. The value of property to a particular investor, or a class of investors, for identified investment objectives. This subjective concept relates specific property to a specific investor, group

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7 See the excerpted passages of Fisher and Williams in EBV Chapter 3 “The Failure of the Neoclassical Investment Rule.” Note how Fisher used the term “investment opportunity” and Williams the term “investment value.”

Not all contemporary authors ignore this distinction. For example, John Hull, in the standard reference *Options, Futures, and Other Derivatives*, distinguishes between “consumption assets” and “investment assets,” cautioning investors about using Finance techniques intended for the latter for the former. See, e.g., Hull (2000, p. 50, 55).
of investors, or entity with identifiable investment objectives and/or criteria.8

Similar definitions in common use include “the value to a specific investor, based on that investor’s requirements, tax rate, or financing,” and “the value of an asset to its owner, depending on his or her expectations and requirements.” These statements implicitly emphasize the distinction between the value in the marketplace of many buyers and sellers (which would establish the fair market value) and the “investment value” to a particular owner.

“Fair Value”

The term “fair value” has a specific meaning when used in financial statements created under Generally Accepted Accounting Principles (“GAAP”) in the United States. These principles are used in reporting financial statements for most public and private firms.9 Under the recently-adopted statement of the body that governs GAAP accounting principles, “fair value” is defined as follows:

Fair value is the price that would be received to sell an asset or paid to transfer a liability in an orderly transaction between market participants at the measurement date.10

The standard goes on to describe an “orderly transaction,” noting that it requires some “exposure to the market” but is not a “forced” sale. It also describes the market to be either the “principal” or “most advantaged” market for the particular asset. “Market participants” are defined as parties that are “independent” of the reporting entity, “knowledgeable” about the asset, and “able” and “willing...but not forced” to complete the transaction.11

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8 Excerpted from the Exposure Draft of IVS2, dated July 2006, section 3.

9 Reporting financial statements are typically not the same as those used for tax purposes, and may not be the same as those used by management or for regulatory purposes. However, they are often the preferred set of statements for investors and managers, because they use a consistent set of rules based on the principle of providing investors with an accurate picture of the enterprise.


11 FAS 157 (September 2006), paragraphs 7-10.
The Financial Accounting Standards Board (FASB) noted that there were differences in the previously-used definitions of “fair value,” which resulted in inconsistent reporting. It also noted that the new definition of “fair value” adopted for reporting purposes was “generally consistent with similar definitions of fair market value used for valuation purposes,” citing the “fair market value” definition adopted by the Internal Revenue Service in RR 59-60.

“Fair Value” Under Specific State Statutes

One specific use of “fair value” arises from the use of the term in the Uniform Business Corporation Act, and is often used in shareholder disputes. The exact meaning of “fair value” in such cases can vary from state to state, and be dependent on case law as well as statute.

Intrinsic Value

The intrinsic value of an asset is its inherent or underlying worth. The idea of “intrinsic value” in philosophy is centuries old, and means the value “in itself” or “for its own sake.” This may be quite different from the market value, which is set by competing buyers and sellers. Thus, the notion of intrinsic value is based on the assumption that an asset can be “worth” more to one set of persons than to another set, and even that society as a whole can mis-estimate the value of something or someone.

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12 See FAS 157, “Summary.” The new statement replaces FAS 141, which had been the basis for many disputes.

13 See FAS 157, paragraph C50.

14 Hitchner (2006, ch. 1) has a nice summary of this topic, along with a comparison to other standards.

15 For example, The Stanford Encyclopedia of Philosophy quotes Plato’s dialogues (about 400 BC), Epicurus (about 300 BC), John Stuart Mill [1806-1873], and Immanuel Kant [1724-1804] on the question of what is intrinsically good. Philosophers have sometimes distinguished this from “extrinsic value.”

16 Contemporary philosophical debates exist, for example, on what societies should do to protect the “intrinsic value” of human life, of other species, of specific ecosystems, and of cultural or historical treasures.
Intrinsic value and investment philosophy

An investor may consider a business to have intrinsic value that is not reflected in the current market price, and therefore may decide not to sell. Other investors may estimate the intrinsic value of stock in certain companies, and purchase stock when they consider the intrinsic value to be higher than the market price. This is the underpinning of the “value” approach to investment popularized by Graham & Dodd. The analysis of the underlying intrinsic value is sometimes called “fundamental analysis.”

Intrinsic value of an option

This term also has a technical meaning pertaining to financial options. For example, a call option is a contract that gives the owner the right, but not the obligation, to buy a security at a certain price during a certain time period. The intrinsic value of a call option is the positive difference (if any) between the market price and the “strike” price of the option.

Other Standards Used in Accounting and Tax

Historical Cost

The historical cost principle is a bedrock upon which accounting practice has developed. See Chapter 6, “Accounting for the Firm.” Most assets used in a business are put on the books at historical cost. If they are not sold, disposed, or transferred to another party, they typically are presented on those statements as worth the historical cost, less any accumulated depreciation or amortization.

Historical cost is not a standard of value; it is an accounting convention. Indeed, all accounting records, taken together, are explicitly not an indication of value—as FASB takes pains to explain:

17 Investors that pursue the philosophy of selecting companies with high underlying intrinsic values relative to their stock prices are sometimes called “value investors.” The approach was first popularized by Benjamin Graham and David Dodd in their 1934 book *Security Analysis*, which has been reprinted and revised many times.

One famous investor that followed this advice was Warren Buffett, who received an “A+” grade from Benjamin Graham 1951 class at Columbia University. His remembrances are included in the 1984 article “The Superintendents of Graham-and-Doddsville.”

Financial accounting is not designed to measure directly the value of a business enterprise, but the information it provides may be helpful to those who wish to estimate its value.19

A similar pronouncement is made in the following statement from an appraisal authority:

In this, value differs from price or cost. Price and cost refer to an amount of money asked or actually paid for a property, and this may be more or less than its value.20

Ubiquity of “Book Value” Term

Despite the clear statements by authorities that cost does not equal value, “book value” is a standard term in business, tax, and finance. Unless it happens by coincidence, most assets do not have a market value equal to the amount shown on an accounting balance sheet as “historical cost less accumulated depreciation.”

Perhaps fortunately, most assets do not need to be bought or sold on a regular basis.21 Therefore, the use of book value, rather than market value, is a useful convention.

Tax Basis

An investor that purchases equity interests in a firm has a basis in that investment. The basis is normally equal to the cost of acquiring it.22 This


21 Indeed, items that are regularly purchased and used are probably not specific assets of a business, but part of its cost of goods sold. What remains at the end of the period is typically inventory of like items.

22 In the US, the IRS defines “basis” as follows in Publication 551 (2002 edition):
The basis of stocks or bonds you buy is generally the purchase price plus any costs of purchase, such as commissions and recording or transfer fees. If you get stocks or bonds other than by purchase, your basis is usually determined by the fair market value (FMV) or the previous owner’s adjusted the basis of stock.

For the purposes of calculating capital gains, there may be adjustments that are necessary, such as for depreciation.
basis is then used to calculate gains or losses for tax purposes when the asset is sold or transferred.

The tax basis is therefore closely associated with the accounting principle of historical cost. It is also not, in general, equal to the fair market value of an asset at any time after it was originally purchased.

Section 1.2. The Human Condition and “Winner’s Curse” in Business Valuation

Many mergers and acquisitions in recent decades have been proposed to investors as value-increasing propositions. One frequently-claimed benefit was the potential for “synergies” that could result from the combination of the two organizations. So many of these failed, however, that it has become a folk theorem that most mergers and acquisitions fail.\(^{23}\) Although it is probably impossible to know whether most mergers and acquisitions fail, there is evidence to support the theorem.\(^ {24}\)

A variation of this folk theorem in mergers & acquisitions is called the “winner’s curse.” Many mergers involve a competition among companies desiring to acquire and control one another, and in a large share of these, the “winner” emerges with a business that subsequently endures a reduction in market value.\(^ {25}\)

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23 Ferris & Pettit (2002, chapter 2) cite unspecified articles in business publications to support this claim:

   Why did more than 50 percent of the major mergers and acquisitions in the United States completed in the 1990s, according to Business Week magazine, erode shareholder value? And why did more than 77 percent of those transactions, according to Forbes magazine, not earn a rate of return at least equivalent to the cost of the capital necessary to finance them? The answer to both questions is often the same: overestimation of target firm value.

24 These include some rigorous research involving samples or transactions (such as Roll [1986], who attributes much of it to “hubris”); the detailed examples of the type included in the Ferris & Pettit book; and other examples that appear, from time to time, in the popular business press. Another possible alternative would be gathering a good sample of well-publicized transactions involving public companies, and compare the subsequent market returns of the stocks involved with those of appropriate market benchmarks.
How could so many highly-skilled corporate titans, investment bankers, lawyers, business strategists, and M&A specialists make decisions that appear to be so poor? What does this say about the human condition, or (much less ambitiously) about valuation methodology?

Let us consider three documented reasons for such a failure, and one additional possibility.

**Three Sound Bases**

Let us assume, as ample evidence suggests—but does not prove—that a very large share of mergers and acquisitions “fail,” meaning that they result in a lower value to stockholders than if the firms had remained separate. There are at least three sound bases for such a phenomenon, all documented in the economics literature and displayed in real life:

1. Auction fever
   The behavior of bidders at auctions underlies the classic “winner’s curse” hypothesis. Auction fever impels many bidders, caught up in the excitement and the competition, to bid higher than some notion of intrinsic worth.
   Such behavior is amply documented in the literature, and commonly on display at auctions. Indeed, in contemporary times, auctions or similar events for such items as art, classic cars, and even antiques are considered entertaining enough to be the subject of television programs.26

2. Hubris
   The hubris of the acquirer was cited as one reason for a winner’s curse by the economist Richard Roll (1986). Hubris—tragic pride—in the

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25 The term “winner’s curse” generally applies to behavior in any auction, and refers to the well-documented propensity of “winning” bidders to pay more than they might, in a more sober mood, want to bid. The term first appeared in the economics literature in an analysis of oil lease bidding by Capen, Clapp, and Campbell (1971), who worked for the Atlantic Richfield energy firm (ARCO). Within the world of corporate M&A, the term found its way into the subtitle of a popular merger & acquisition text: *Valuation: Avoiding the Winner’s Curse* [Ferris & Pettit (2002)]. See note above.

26 To be sure, this is a very low bar. However, the entertainment aspect indicates that human emotion is driving behavior, creating a drama that captivates at least some viewers completely uninterested in the financial aspects of the auction.
human condition was examined by many of the great ancient Greek writers, as well as Shakespeare, Dante, and numerous others.\textsuperscript{27} The centuries have not banished it. It certainly exists in the business world, as well as in politics, entertainment, sports, sibling rivalry, and the local homeowner’s association.

3. The synergy trap

Organizations are not simple additions of asset values. Many mergers and acquisitions involve synergistic elements, which may distort comparability to the subject company. In general, the whole of two organizations can be more, or less, than the sum of the two parts. This is especially the case when the two are forcibly integrated.

Thus, mergers that promise “synergy” savings can result in costs and performance deterioration, as well as confusion among customers and employees, that overwhelm whatever actual savings occur.

A Modest Suggestion

It maybe unwise to deny the role that hubris plays in mergers & acquisitions, or to ignore the evidence of auction fever. Thus, there are some anomalies in economic behavior on display.\textsuperscript{28} However, it is also unwise to ignore the possibility that many highly-skilled financial practitioners, with state-of-the-art models and ample computer resources and data, simply got it wrong.

Perhaps some of their errors stem from defects in their valuation methodologies. This is certainly part of the problem: Ferris & Pettit (2002), along with others argue that using traditional DCF tools, but doing a better job, would have avoided at least some of these errors.\textsuperscript{29}

A Deeper Critique

- There is, however, a deeper critique. Viewing companies as homogeneous entities, which can be combined and separated without damage

\textsuperscript{27} Ancient Greek literature elevated \textit{hubris} to a central element of such epics as Homer’s \textit{Iliad} and \textit{Odyssey}. Dante, in the \textit{Inferno}, relegated sinners guilty of betrayal and pride to the lowest level of hell. Hubris is dramatized in various Shakespeare plays including \textit{Twelfth Night} (“Be not afraid of greatness: some are born great, some achieve greatness, and some have greatness thrust upon ‘em”); \textit{Othello}, \textit{Macbeth}, \textit{Julius Caesar} and others.

\textsuperscript{28} The field of Behavioral Finance focuses on these anomalies. Thaler (1988) is an early examination of anomalies in auction behavior and the existence of a winner’s curse.
or benefit except to efficiencies of operation, is fundamentally wrong. If we return to the definition of the firm stated in EBV Chapter 4 “The Nature of the Firm,” we recognize this in the first few words: a firm is an organization, and an organization consists of people. People, to say the obvious, are not homogeneous. Thus, if the “whole” does not take into account the essential attributes of the firm, how can the “parts”?

29 In particular, Ferris & Pettit (2002, chapter 2) describe a “ROE” [return on equity] model, in which the accounting ratios that drive earnings to shareholders are modeled and forecast into the future.

We discuss models of this type in EBV Chapter 13 “Traditional Valuation Methods” and Chapter 14 “Practical Application of the Income Method.”
Chapter 2

The Nature of the Firm

by Samantha Superstine,

with contributions from Patrick L. Anderson

An Introduction

We discuss in The Economics of Business Valuation Chapter 4, “The Nature of the Firm,” how the presentation of the firm in both standard microeconomics and macroeconomics is still quite primitive. In microeconomics, firms are typically assumed to sell homogenous goods, using a simple production function; markets clear on the profit-maximizing criteria of one-period firms arrayed against aggregate consumer demand in the market; workers consider their wages and available interest rates, and adjust their consumption plans accordingly.30 Additionally, the driver of businesses (the entrepreneur) is largely ignored in mainstream economics.31

This chapter is meant to supplement EBV Chapters 4, 5, and 6. We begin by discussing the principal-agent and moral hazard problems in firms. Then we use business data to compare U.S. firms by size and type of firm. We then provide a discussion on business entities as they are classified into different types of categories based on tax filing status (C corporations, S corporations, partnerships, etc). Lastly, we discuss employment and establishment survival rates in the U.S.

30 We describe this more fully in EBV Chapter 8, “Value in Neoclassical Economics.”
31 The entrepreneurial component of new businesses is gaining attention in the mainstream media and even academia. However, the role of the entrepreneur has not been extensively studied, except for in Austrian economics. The editor recommends for the curious reader the following: Israel Kirzner’s Competition and Entrepreneurship, Carl Menger’s Principles of Economics, Peter Klein’s The Capitalist and The Entrepreneur, and Peter Boettke’s Living Economics: Yesterday, Today, and Tomorrow.

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SECTION 2.1. Principal-Agent, Moral Hazard, and Other Problems

Unfortunately, the principal-agent problem in business (and related moral hazard and other problems) is not solely theoretical, and has produced tangible damage over the past century. In this subsection, we discuss two seminal periods in which public attention was focused on these issues.

Public Concern over Principal-Agent Problems: the 1930’s

Public concern over separating the ownership of and control over business ventures arose centuries after Adam Smith’s statement of the problem. One influential, if alarmist, view was propounded by lawyer Adolph Berle and economist Gardiner Means in *The Modern Corporation and Private Property*. First published in 1932, it stated that the separation of ownership and control in the modern corporation would cause many social ills. Those that control the corporation, they wrote, can “serve their own pockets better by profiting at the expense of the company than by making profits for it.”32 This is an acute implication of “agency theory,” although this phrase makes no appearance in the book, for it had not yet been propounded.33

There are two other, more subtle observations of Berle & Means that deserve note. First, they observed the separation of property into consumption property and productive property. The latter category includes investments: if investments are a different type of “property” than the traditional home-and-hearth property, they contend, their owners will behave differently toward them. Second, they contend that separating ownership from control undermines the profit maximization principle itself.34

33 The Introduction to the 1991 Transaction Edition, written by Murray Weidenbaum and Mark Jensen, contains a balanced discussion about the contributions of the Berle and Means in 1932, the subsequent business and regulatory events (including the disappearance of many former corporate giants, and periodic merger waves), and the later appearance of agency theory.
34 On these points, see the Introduction to the 1991 Transactions Edition, and the passages in Book IV, Chapter 2, and Book I, chapter 6.
Recent Accounting Scandals

Decades later, public concern again arose over widespread abuse, underlining the seriousness of the principal-agent problem. Ironically, it was the perversion of one of the measures from the 1930’s that led to this wave of reform: the use of an “independent” accounting firm to audit the books.

Indeed, the word “public” in the term “certified public accountant” highlights the duty to the public interest that is supposed to be served when an independent accountant prepares or reviews accounting records. In most cases, accounting firms play a valuable and specific role as the independent auditor to publicly-traded firms.

Over time, however, these same firms learned they could also act as consultants to the company. In such a role, they could earn very large fees for advice on a range of managerial, accounting, tax, and technical issues. The boards of directors of such companies should have identified this conflict of interest and taken steps to prevent abuses. Unfortunately, in some cases the fees earned from these consulting arrangements corrupted the independence of the audit function performed by the same firm. This was clearly evidenced in scandals such as those involving Enron Corporation, Worldcom, and others. However, the extent of the conflicts of interest extended far beyond these firms. Tirole (2006, section 1.2) notes that, as of 2001, nonaudit fees constituted more than half of the fees paid to accounting firms in 28 of the 30 firms comprising the Dow Jones Industrials.

The damage from these scandals went far beyond the destruction of a few firms and several careers. As investors became aware of the corruption of the role of “independent” auditing by accounting firms, they began to worry about the accuracy of financial reports for publicly traded companies in general. Due to these concerns, an enormous amount of market capitalization of publicly-traded firms evaporated. Largely as a result of these scandals, Title II of the federal Sarbannes-Oxley Act of 2002 now prohibits the use of auditors to perform significant consulting tasks for the same firms.35

35 Sarbannes-Oxley Act of 2002, Title II, section 102. Of course, as in most laws there are exceptions and other things? variables? that managers should consider before concluding that a specific arrangement is lawful or unlawful.
There is nothing uniquely American in this experience; a number of other countries have experienced similar accounting scandals (though few on the scale of those in the U.S.). As a result of these and other concerns, many countries have adopted codes of good governance, including Brazil, France, Russia, the UK, and Singapore. These codes supplement other rules, including stock exchange rules, state and national laws, and accounting standards.

**Implications: Investor Confidence, Information, and the Principal-Agent Problem**

The passage of a law rarely solves a fundamental problem, and Sarbannes-Oxley is no exception. The fundamental principal-agent problem still exists with large businesses in which stockholders do not manage the firm. Although auditor independence is now better than before, many firms still rely on accountants to perform a range of services outside of auditing. Furthermore, investors rely on information that is not the responsibility of a firm’s accountants, such as management’s assessments of business conditions, competitor actions, and the company’s pipeline of new products. These statements to the investment community are fundamentally the decisions of managers, and it is their responsibility to ensure any ensuing statements are based on good information, whether the statements originate from accountants, engineers, economists, financial managers, operational experts, lawyers, or public relations consultants.

**SECTION 2.2. Comparing U.S. Firms by Size**

Do very large, publicly-traded firms dominate the U.S. economy (meaning that their revenues, employees, profits, and influences exceed that of other firms)? Do they tower so high above the teeming masses of small businesses that the American economy depends more on a handful of big corporations than on smaller companies?

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36 See Tirole (2006, section 1.3).
37 Recall that the Sarbannes-Oxley proscription only applies to firms covered by the Act, which are largely publicly-traded firms.
We present data below, which indicate that the businesses that employ a majority of private-sector workers, are neither large nor publicly-traded. In particular, the data establishes that the majority of the employment and earnings in the private sector economy of the United States come from small and mid-sized firms, almost all of which are privately held.38 Furthermore, a significant fraction of very large firms are privately held.

a. The Role of Small and Large Firms

As stated in EBV Chapter 4 “The Nature of the Firm,” small businesses (which we define as nonfarm entities with fewer than 500 employees) are a significant part of the national economy. While it may be easy to assume that large firms with recognizable names and products are responsible for the majority of the economy, small businesses account for at least 40% of national numbers for key economic indicators.

In 2010, small businesses accounted for 49% of employment, 43% of annual payroll figures, and 84% of establishments nationwide.39 To extend these calculations to firms, small businesses make up 99.7% of firms nationwide, leaving large businesses to account for only 0.3% of firms. In other words, of the 5.7 million firms in 2010, only slightly more than 17,000 employed more than 500 workers.

Despite the ratio of small to large businesses, large and midsized firms are responsible for most business revenue in the United States. In 2007, large firms accounted for 62% of revenues ($18.4 trillion of approximately $29.7 trillion), with the rest coming from small and midsized firms.40

38 The U.S. Census Bureau’s County Business Patterns (CBP) covers data on most industries, excluding crop and animal production; rail transportation; National Postal Service; pension, health, welfare, and vacation funds; trusts, estates, and agency accounts; private households; and public administration. More details about the data used for CBP data can be found here: http://www.census.gov/econ/cbp/overview.htm

39 The U.S. Census Bureau distinguishes between “firms” and “establishments.” An establishment is a fixed location at which economic activity occurs, and therefore a firm may have one or many establishments. “Businesses” may be used referring to either type of entity. Establishment activity is defined by March employment, and excludes governmental establishments except liquor stores, wholesale liquor establishments, depository institutions, federal and federally sponsored credit agencies, and hospitals.

40 Most recent revenue data is only available every 5 years. See Table 2-1 on page 19 for 2007 business data from the U.S. Census.
See Table 2-1 and Table 2-2 below for detailed statistics on employment, payroll, and when applicable, revenues, by firm size. For every metric since 2007, the large business category has had slight increases in its contribution to national figures.

### TABLE 2-1. Firm Data by Size (2007)

<table>
<thead>
<tr>
<th>ENTERPRISE EMPLOYMENT SIZE</th>
<th>NUMBER OF FIRMS (a)</th>
<th>NUMBER OF ESTABLISHMENTS</th>
<th>EMPLOYMENT (b)</th>
<th>ANNUAL PAYROLL ($1000s)</th>
<th>ESTIMATED RECEIPTS ($1000s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-4</td>
<td>3,705,275</td>
<td>3,710,700</td>
<td>6,139,463</td>
<td>$234,921,325</td>
<td>$1,434,680,823</td>
</tr>
<tr>
<td>5-9</td>
<td>1,060,250</td>
<td>1,073,875</td>
<td>6,974,591</td>
<td>$222,419,546</td>
<td>$1,144,930,232</td>
</tr>
<tr>
<td>10-19</td>
<td>644,842</td>
<td>682,410</td>
<td>8,656,182</td>
<td>$292,088,277</td>
<td>$1,395,498,431</td>
</tr>
<tr>
<td>&lt;20</td>
<td>5,410,367</td>
<td>5,466,985</td>
<td>21,770,236</td>
<td>$749,429,148</td>
<td>$3,975,109,486</td>
</tr>
<tr>
<td>20-99</td>
<td>532,391</td>
<td>723,385</td>
<td>20,922,960</td>
<td>$768,546,555</td>
<td>$3,792,920,977</td>
</tr>
<tr>
<td>100-499</td>
<td>88,586</td>
<td>355,853</td>
<td>17,173,728</td>
<td>$686,862,018</td>
<td>$3,612,050,221</td>
</tr>
<tr>
<td>&lt;500</td>
<td>6,031,344</td>
<td>6,546,223</td>
<td>59,866,924</td>
<td>$2,204,837,721</td>
<td>$11,380,080,684</td>
</tr>
<tr>
<td>500+</td>
<td>18,311</td>
<td>1,158,795</td>
<td>60,737,341</td>
<td>$2,821,940,511</td>
<td>$18,366,661,220</td>
</tr>
<tr>
<td>Total</td>
<td>6,049,655</td>
<td>7,705,018</td>
<td>120,604,265</td>
<td>$5,026,778,232</td>
<td>$29,746,741,904</td>
</tr>
</tbody>
</table>

- Percentage for <500: 99.7%
- Percentage for 500+: 0.3%

### TABLE 2-2. Firm Data by Size (2010)

<table>
<thead>
<tr>
<th>ENTERPRISE EMPLOYMENT SIZE</th>
<th>NUMBER OF FIRMS (a)</th>
<th>NUMBER OF ESTABLISHMENTS</th>
<th>EMPLOYMENT (b)</th>
<th>ANNUAL PAYROLL ($1000s) (c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-4</td>
<td>3,575,240</td>
<td>3,582,826</td>
<td>5,926,452</td>
<td>$226,541,056</td>
</tr>
<tr>
<td>5-9</td>
<td>968,075</td>
<td>982,019</td>
<td>6,358,931</td>
<td>$212,039,611</td>
</tr>
<tr>
<td>10-19</td>
<td>617,089</td>
<td>652,662</td>
<td>8,288,385</td>
<td>$283,246,473</td>
</tr>
<tr>
<td>&lt;20</td>
<td>5,160,404</td>
<td>5,217,507</td>
<td>20,573,768</td>
<td>$721,827,140</td>
</tr>
<tr>
<td>20-99</td>
<td>475,125</td>
<td>648,386</td>
<td>18,554,372</td>
<td>$719,061,251</td>
</tr>
<tr>
<td>100-499</td>
<td>81,773</td>
<td>354,313</td>
<td>15,868,540</td>
<td>$665,644,629</td>
</tr>
<tr>
<td>&lt;500</td>
<td>5,771,302</td>
<td>6,220,206</td>
<td>54,996,680</td>
<td>$2,106,533,020</td>
</tr>
<tr>
<td>500+</td>
<td>17,236</td>
<td>1,176,422</td>
<td>56,973,415</td>
<td>$2,834,450,349</td>
</tr>
<tr>
<td>Total</td>
<td>5,734,538</td>
<td>7,396,628</td>
<td>111,970,095</td>
<td>$4,940,983,369</td>
</tr>
</tbody>
</table>

- Percentage for <500: 99.7%
- Percentage for 500+: 0.3%

(a) The U.S. Census Bureau distinguishes between “firms” and “establishments.” An establishment is a fixed location at which economic activity occurs, and therefore a firm may have one or many establishments.
(b) Employment is defined by the number of employees on the payroll at the time of data collection (Mid-March pay period)
(c) Receipt data available every five years; most recent release is 2007 data.

Analysis: Anderson Economic Group, LLC
Number of Firms, Establishments, and Employees for Large Businesses

Noting the vague boundaries between “midsized” and “large” firms, it is difficult to neatly compare the economic significance of each. However, by issuing a specific cutoff by which to define a boundary between midsized and large firms, we can estimate the dominance of large corporations with a bit more precision.

Having looked at the contributions of different-sized firms to the national economy, we can now turn to data that is specific to large business to analyze what portion of national employment and payroll come from businesses that are considered to be “large enterprise size.”

We can roughly classify firms with more than 5,000 employees as “big corporations”; firms with 5,000 or less employees can therefore be classified as small and midsized firms. EBV Chapter 5 “The Organization and Scale of Private Business” discusses in further detail the reasoning behind classifying certain levels of employee size as “small,” “midsized,” or “large.”

Table 2-3 on page 21 displays firms, establishments, and employees for large businesses in 2007 and also offers a comparison to small and midsized firms as a whole. Table 2-4 on page 22 shows the same data for 2010, with the exception of business receipts. The proportion of employment and payroll between the two time periods is very similar, with less than a percentage point decrease in the contribution of small and midsized firms from 2007 to 2010.

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41 As stated in EBV Chapter 5, large enterprise can be defined as firms with greater than 5,000 employees.
42 Data on estimated receipts is released every five years, with the last release being for 2007 data.
The 2010 data show that small and midsized firms make up 67% of private-sector employment, and 62% of private-sector payroll. “Big corporations” are not responsible for the majority of national employment and earnings in the private sector. These statistics further establish the percentage of the nation’s workers that depend on small and midsized firms for employment.

43 2007 data also indicate that small and midsized firms account for 57% of revenues for private-sector firms. More recent data is not available for receipts.
Section 2.3. Classifying Ownership of Businesses

The data in the previous section implies that the notion of dominant large corporations is a myth, if you define “large corporations” as those that are numbered on either of the Fortune 1000 or the Forbes Global 2000 lists. If these data correctly portray the U.S. economy, privately-held firms should be identified as the mainstay of the U.S. economy.

Business entities can be classified into different types of categories based on tax filing status, which we discuss in greater detail in Section 2.4., "U.S. Firms by Corporation Type” on page 28. These categories, however, are associated with how each entity is owned. This section provides insight into variations in types of businesses, based on the ownership of each separate entity. Detail regarding the characteristics of publicly-traded, privately-held, or non-firm/non-business entities is shown in Exhibit 2-1, “Characteristics of Filing Entities,” on page 27. Select types of variations are explained below.

a. Publicly-Traded Corporation Variations

Publicly-traded companies allow for partial ownership to be acquired via purchases and sales by the public. As depicted in Exhibit 2-1, “Charac-
characteristics of Filing Entities,”, publicly-traded companies include many C corporations (C corps). There are other forms of public corporations that should be considered:

1. Mutual Funds
   A mutual fund, in particular a stock mutual fund, is an investment fund that is organized by investors who purchase shares in the fund, with the fund using those proceeds to purchase equity investments in other publicly-traded firms, along with cash investments and other securities. In general, we consider these to be pass-through entities, allowing private investors to achieve some diversification and professional investment management, primarily for their investments in publicly-traded firms. However, some mutual funds (such as “hedge funds” and “private equity” funds) can invest in private firms.

2. Real Estate Investment Trusts (“REITs”)
   These trusts invest in real estate, and allow for special tax treatment of real estate to flow through to investors. REITs can be part of more complicated structures, including publicly-traded companies. Taubman Centers is an example of a publicly-traded corporation whose principal operating business is a REIT.

b. Privately-Held Variations

Privately-held corporations include some C corps, as well as most S corporations (S corps) and partnerships. Through the centuries, there have been an enormous variety of privately-held business organizations. We describe in some detail the most important in terms of economic scale and taxable activity in Exhibit 2-1, “Characteristics of Filing Entities,” on page 27. However, there are some other variations that deserve note:

1. The profitable association
   Associations often are founded for charitable, social, and benevolent purposes. After some years, these same associations may begin to form business entities to serve their members. In a number of cases, these business entities have grown quite large.
   Life, health, and auto insurance; retirement housing; drug purchasing; affiliated merchandise sales; and land development are some of the industries in which these business entities provide goods and services. Depending on the profit motive and separate identity, these organizations may or may not constitute *bona fide* firms.
2. The criminal syndicate

The history of 20th-century America includes many chapters of criminal gangs, often originating in various ethnic emigrations to the United States. A number of these grew to become business operations that appear to meet the definition of a firm. However, the notion of a separate legal identity implies that ownership interests can be purchased and are entitled to the protections of law; these conditions are probably not met for criminal syndicates.

3. The government-sponsored enterprise

In the United States, there are a number of government-sponsored enterprises (“GSEs”) that operate in a manner similar to a business. Under the federal law that initially defined them, these entities have private stockholders or other equity owners, and a board of directors which may be elected by private owners. However, they also have a government charter and therefore benefit from implied or explicit government guarantees, and often receive subsidies.

There are other variations on the quasi-public organization. These include the United States Post Office; the Tennessee Valley Authority; and a large variety of prisoner-labor undertakings. There are also agencies of state and federal governments that attempt to operate like a private firm in some manner. Some of these are covered in the category listed below.

We do not consider these to be “firms” because they fail at least the profit-motive-for-investors test. They may also fail the separate legal identity test, because they carry an implied government guarantee. Sometimes they operate within a government-protected market as well.

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44 They had a separate legal identity, they had a profit motive for the “investors,” and they had replicable “business” practices.

45 A 2007 Congressional Research Service memo lists the federal GSEs as follows:

- Three of the GSEs — the Federal National Mortgage Association (Fannie Mae), the Federal Home Loan Mortgage Corporation (Freddie Mac), and the Federal Agricultural Mortgage Corporation (Farmer Mac) — are investor owned; the others — the Federal Home Loan Bank System and the Farm Credit System — are owned cooperatively by their borrowers.
- The same memo lists technical reasons why the Financing Corporation and the Resolution Financing Corporation should not be included, as well as stating that the former GSE known as “Sallie Mae” changed into a private corporation in 2004.

46 Congress defined the term “government-sponsored enterprise” in the Omnibus Reconciliation Act of 1990.

47 The 2007 CRS memo reports a 2000 CBO estimate of the borrowing subsidy of specific federal GSEs at $13.5 billion.
4. The government-chartered enterprise

In addition to those entities that are sponsored by state or federal governments, there are also a significant number of business organizations that may have been chartered by a government at their inception, or that receive a specific charter during their operation, but are not fully sponsored by a government. By “charter” we mean a grant, charter, or statute that identifies a purpose, and also provides certain authorities and benefits. Often, the benefits include exemptions from certain taxes that private businesses must pay. These organizations, however, are expected to compete with other businesses in at least some markets. They may also assert that they are profit-motivated companies, and are often organized in a similar manner. They may generate profits, and such profits could be used to purchase other firms.

Examples of these include some large health-care organizations such as “Blue Cross” affiliates in some states, and certain unemployment insurance or workers’ compensation providers. The question of whether such an organization is fully sponsored, or merely chartered, by the government is often a delicate one. Furthermore, the benefits the organizations receive—notably tax exemptions—are often controversial.\textsuperscript{48}

The question of whether such organizations are firms, and whether they would have value without the special government charter, requires careful study of the actual business, the government-provided benefits, and any government-imposed burdens.

Imputed Earnings in Pass-Through Entities

A number of business forms result in the taxable earnings of the business being imputed to the equity owners. These equity owners may be called shareholders, members, partners, or some other term. If the firm’s earnings are imputed for tax purposes to the equity owners, and the firm itself does not pay income taxes on its earnings, we call it a pass-through entity.

\textsuperscript{48} For example, many of the Blue Cross Blue Shield entities were exempt from federal taxes for most of the 20th century, under section 510(c)4 of the Internal Revenue Code. However, Congress revoked the tax-exempt status for most “nonprofit” health insurers in 1986; section 510(m) now explicitly taxes “commercial-type insurance” even if it is provided by a nonprofit entity. Some states (such as Michigan) allow a state tax exemption to such firms, others do not. See Sallee et al. (2007, section II).
For example, a firm that earned (under tax accounting) $100,000 in a year would apportion those earnings to its members; a 50% equity owner would then pay federal income tax on $50,000 in earnings. It is important to note that the income is imputed to the members, rather than actually paid to them. In general, the amount of taxable earnings imputed to members is different, and often quite different, from the amount of money actually distributed to them. Even a firm with steady earnings and capital investments would have taxable earnings different than actual cash earnings or accounting cash flow in any given year. Furthermore, most growing firms must have a portion of their earnings retained in the firm for “working capital” purposes. We discuss this important, and often overlooked, factor in EBV’s Chapter 14 “Practical Application of the Income Method.”
### Exhibit 2-1. Characteristics of Filing Entities

**March 29, 2013**

<table>
<thead>
<tr>
<th>Type of Entity, by Tax Filing Status (a)</th>
<th>Ownership</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Non-firm or non-business entity</td>
</tr>
<tr>
<td>Corporation Filers</td>
<td>relatively few with operating income</td>
</tr>
<tr>
<td>C Corporations (1120)</td>
<td>relatively few with operating income</td>
</tr>
<tr>
<td>S Corporations (1120S)</td>
<td>pass-through entities for real estate and investment earnings</td>
</tr>
<tr>
<td>Investment trusts: REITs, RICs (1120-REIT, 1120-RIC)</td>
<td>some; excluded from &quot;nonfinancial&quot; firms</td>
</tr>
<tr>
<td>Financial Institutions (Banks, Insurance Companies)</td>
<td></td>
</tr>
<tr>
<td>Partnerships (b)</td>
<td>relatively few with operating income</td>
</tr>
<tr>
<td>Limited and General (1065)</td>
<td>relatively few with operating income</td>
</tr>
<tr>
<td>LLCs (1065)</td>
<td></td>
</tr>
<tr>
<td>Sole Proprietorships</td>
<td>almost all</td>
</tr>
<tr>
<td>Sole proprietors (Schedule C)</td>
<td></td>
</tr>
<tr>
<td>Farms and Financial Institutions</td>
<td>most have substantial wage and salary share of revenue; many SP filers are contract workers</td>
</tr>
<tr>
<td>Self-employed farmers (Schedule F); Other Agricultural, Fishery, Timber</td>
<td></td>
</tr>
<tr>
<td>Financial Institutions (banks, thrfts, CUs, insurance companies)</td>
<td></td>
</tr>
<tr>
<td>Other</td>
<td>intended as passive investment vehicle</td>
</tr>
<tr>
<td>Mutual funds holding corporate equities</td>
<td></td>
</tr>
<tr>
<td>Cooperative entities (1120C), Mutual associations</td>
<td></td>
</tr>
<tr>
<td>most, though may have business affiliates</td>
<td></td>
</tr>
<tr>
<td>Nonprofit organizations (Form 990)</td>
<td>some, almost all</td>
</tr>
<tr>
<td>Tax-exempt financial and insurance institutions (e.g. Credit Unions, nonprofit health insurers)</td>
<td></td>
</tr>
<tr>
<td>Charities, Churches, Schools</td>
<td>most, may or may not report taxable business activity</td>
</tr>
<tr>
<td>Informal businesses</td>
<td>some mixture of criminal, informal, wage and salary, and business activity</td>
</tr>
<tr>
<td>Criminal syndicates</td>
<td>some, though may have business affiliates</td>
</tr>
<tr>
<td>State and Local Governments</td>
<td>some, though not filing as a business</td>
</tr>
<tr>
<td>Government Sponsored Entities</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(a) Tax forms listed are based on author's analysis of common filing requirement for category. Actual filing requirements will vary by entity, year, and with changes in tax laws.

(b) Some Partnerships file as S Corporations

Source: Categorization by Author; tax filing information, IRS.
Analysis: Anderson Economic Group, LLC
SECTION 2.4. U.S. FIRMS BY CORPORATION TYPE

Exhibit 2-1, “Characteristics of Filing Entities,” on the previous page explains the tendency of various types of corporations to be publicly-owned, privately-held, or non-business entities. The following section delves deeper into these different types of corporations, first establishing the difference between employer and non-employer firms, and then looking to data sources to assess the contributions these corporations make on the national business economy.

a. Number of Business Entities by Corporation Type

Most data make a distinction between employer and non-employer firms. As discussed in EBV Chapter 5 “The Organization and Scale of Private Business,” employer firms have at least one employee during the year, whereas some tax-paying entities have no paid employees.

The data from the Small Business Administration (SBA) indicate that there are nearly 29 million establishments in 2008, 5.9 million of which had employees. For comparison, IRS data show “business” tax returns filed to be almost 32 million. This is consistent with the theory that companies may file more than one return for business dealings, which is also explained in EBV Chapter 5.

The data also show the number of returns filed by different corporation types. As shown in Table 2-2 on page 29, the number of returns filed by C corps, S corps, and partnerships far exceeds the number of firms reported by the U.S. Census Bureau. The total number of filings in 2008 from S corps, C corps, partnerships, and sole proprietorship totals 22.6 million.

Table 2-2 on page 29 displays the reported number of business entities in the U.S., as recorded by several different agencies. Note that the number of firms reported by the SBA is not equal to IRS filings, and that these figures also do not perfectly respond to data presented by the U.S. Census Bureau.

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49 SBA refers to employer firms as well as establishments, listing 5.93 million firms and 7.6 million establishments in 2008. Non-employers were listed at 21.4 million, and self-employment at 10.1 million.
TABLE 2-2. Reported Number of Business Entities in the U.S.

<table>
<thead>
<tr>
<th>Small Business Administration Data</th>
<th>millions</th>
<th>2008</th>
</tr>
</thead>
<tbody>
<tr>
<td>Employer Firms</td>
<td>5.93</td>
<td></td>
</tr>
<tr>
<td>Self-employment (nonincorporated)</td>
<td>10.08</td>
<td></td>
</tr>
<tr>
<td><strong>Sum of Firms</strong></td>
<td>16.01</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Establishments (a)</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-employers</td>
<td>21.35</td>
<td></td>
</tr>
<tr>
<td>Employers</td>
<td>7.60</td>
<td></td>
</tr>
<tr>
<td><strong>Total number of establishments</strong></td>
<td>28.95</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>IRS Data</th>
<th>millions</th>
<th>2008</th>
</tr>
</thead>
<tbody>
<tr>
<td>C Corporation Returns Filed</td>
<td>1.78</td>
<td></td>
</tr>
<tr>
<td>S Corporation Returns Filed</td>
<td>4.05</td>
<td></td>
</tr>
<tr>
<td>Partnership Returns Filed</td>
<td>3.15</td>
<td></td>
</tr>
<tr>
<td>LLCs</td>
<td>1.90</td>
<td></td>
</tr>
<tr>
<td>Other (REIT)</td>
<td>0.02</td>
<td></td>
</tr>
<tr>
<td>Non-Farm Sole Proprietorship Returns Filed in</td>
<td>22.61</td>
<td></td>
</tr>
<tr>
<td><strong>Sum of “Business” Tax Returns Filed in 2002</strong></td>
<td>31.61</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>County Business Patterns Data (b)</th>
<th>millions</th>
<th>2008</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Firms</td>
<td>5.93</td>
<td></td>
</tr>
<tr>
<td>Number of Establishments</td>
<td>7.60</td>
<td></td>
</tr>
<tr>
<td>Payroll (millions)</td>
<td>$5,130,509</td>
<td></td>
</tr>
<tr>
<td>Payroll: 500+ employees</td>
<td>$2,901,341</td>
<td></td>
</tr>
<tr>
<td>Share of payroll from firms with 500+ employees</td>
<td>56.6%</td>
<td></td>
</tr>
</tbody>
</table>

(a) The SBA defines a "firm" as “an aggregation of all establishments (locations with payroll in any quarter) owned by a parent company.”
(b) CBP data are not directly comparable with IRS or SBA data

Analysis: Anderson Economic Group, LLC

b. Estimated Taxable Business Earnings of U.S. Corporations, Partnerships, and Sole Proprietorships

IRS data breaks down business returns by corporation type. For 2008 data,
- Total business net earnings for 2008 were about $1.8 trillion, on $30.4 trillion in business receipts;

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50 EBV Chapter 5 “The Organization and Scale of Private Business” details the definitions for different types of corporations, namely public and private companies, which further consist of C corps, S corps, Limited Liability Companies (LLC), Partnerships, and Sole Proprietorships. Furthermore, Exhibit 2-1, “Characteristics of Filing Entities,” on page 27 details characteristics of different types of filing entities by their propensity to be public, private, or other.
Even though C corps had the largest amount of business receipts, partnerships accounted for the highest net income of all entity types, at $458 billion; and

Sole proprietorships accounted for the least net income, at $265 billion.

Table 2-3 details the business receipts and net income by corporation type for 2003 and 2008. The only values to decrease between 2003 and 2008 are net income reported for C corps and sole proprietorships, which declined by 27% and 1.9%, respectively, over the five-year period. Related to that, the deficit reported by C corps, when adjusted for inflation, increased by 81% between 2003 and 2008, from $380 billion to $690 billion.

All other values saw increases, with the changes being relatively larger for business receipts for every corporation type other than S corps, in which case the percentage change in net income was slightly higher. Sole proprietorships saw the smallest increases over the period of analysis, increasing business receipts by 7.2% and decreasing net income by 1.9%.

| TABLE 2-3. Business Receipts and Net Income by Corporation Type (billions, 2008 dollars) |
|---|---|---|---|---|
| Entity Type | 2003 Business Receipts | Net Income (c) | 2008 Business Receipts | Net Income | Change Business Receipts | Net Income |
| C corps | $16,512 | $533 | $18,705 | $389 | 13.3% | -27.1% |
| S corps | $4,859 | $250 | $6,013 | $317 | 23.8% | 26.8% |
| Partnerships (a) | $2,979 | $353 | $4,344 | $458 | 45.8% | 29.9% |
| Sole Proprietorships (b) | $1,229 | $269 | $1,317 | $265 | 7.2% | -1.8% |
| REIT/RIC | $0 | $179 | $0 | $356 | 0.0% | 98.6% |
| Total | $25,579 | $1,584 | $30,379 | $1,784 | 18.8% | 12.6% |

(a) Partnerships includes general, limited, and LLCs
(b) Nonfarm
(c) Net income less deficit

Source: U.S. Department of the Treasury, Internal Revenue Service, SOI Bulletin
Analysis: Anderson Economic Group, LLC

As indicated above, C corps had the most business receipts, while partnerships generated the highest amount of net income in 2008. Some of these firms are publicly held, and some are privately owned. From this basis, using some professional judgment, we can allocate earnings among privately-held and publicly-traded firms, as well as allocate earnings that we categorize implicitly as wage and salary earnings reported on business
tax returns. Furthermore, we can note that corporation types such as REIT/RIC are classified as non-business entities, reporting no business receipts to the IRS.

Table 2-4 shows estimates of taxable earnings for entity type, differentiating between net income contributed by private and public firms, as well as non-business entities. These estimates indicate that more than half (55%) of net income is generated by privately-held firms, and only 15% by publicly-held firms.

**TABLE 2-4. Estimated Taxable Earnings of U.S. Filing Business Entities**

(thousands, 2008 dollars)

<table>
<thead>
<tr>
<th>Type of Organization</th>
<th>Reported Business Receipts</th>
<th>Reported Net Income (a)</th>
<th>Estimated Net Income by Type of Organization (b)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Non-business</td>
</tr>
<tr>
<td>C corps</td>
<td>$18,704,829</td>
<td>$388,740</td>
<td>$ -</td>
</tr>
<tr>
<td>non-business</td>
<td>0%</td>
<td>70%</td>
<td></td>
</tr>
<tr>
<td>public</td>
<td>0%</td>
<td>30%</td>
<td></td>
</tr>
<tr>
<td>S corps</td>
<td>$6,013,292</td>
<td>$317,091</td>
<td>$ -</td>
</tr>
<tr>
<td>non-business</td>
<td>0%</td>
<td>100%</td>
<td></td>
</tr>
<tr>
<td>public</td>
<td>0%</td>
<td>0%</td>
<td></td>
</tr>
<tr>
<td>REIT/RIC (c)</td>
<td>$ -</td>
<td>$355,576</td>
<td>$355,576</td>
</tr>
<tr>
<td>non-business</td>
<td>100%</td>
<td>0%</td>
<td></td>
</tr>
<tr>
<td>public</td>
<td>0%</td>
<td>0%</td>
<td></td>
</tr>
<tr>
<td>Partnerships (d)</td>
<td>$4,343,871</td>
<td>$458,185</td>
<td>$ -</td>
</tr>
<tr>
<td>non-business</td>
<td>0%</td>
<td>0%</td>
<td></td>
</tr>
<tr>
<td>public</td>
<td>0%</td>
<td>100%</td>
<td></td>
</tr>
<tr>
<td>Sole Proprietorships (e)</td>
<td>$1,317,443</td>
<td>$264,508</td>
<td>$171,930</td>
</tr>
<tr>
<td>non-business</td>
<td>65%</td>
<td>0%</td>
<td></td>
</tr>
<tr>
<td>public</td>
<td>35%</td>
<td>0%</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>$30,379,436</td>
<td>$1,784,100</td>
<td>$527,507</td>
</tr>
</tbody>
</table>

### Share of total net income

|                      | 30% | 15%  | 55%  |

(a) Net income less deficit
(b) Estimates of apportionment to public/private organizations for C corps and Sole Proprietorships based on professional judgment
(c) Although REIT/RIC corporations may be publicly-owned or privately-held, transactions are considered to be non-business
(d) Partnerships includes general, limited, and LLCs
(e) Nonfarm

Source: Internal Revenue Service, SOI Bulletin
Analysis: Anderson Economic Group, LLC
SECTION 2.5. Firm Survival

Businesses are constantly entering and exiting the economy. Now that we have looked at businesses broken down by corporation and ownership type, we can reaggregate data to gain insight into patterns of firm survival rates. Survival rates are influenced by many factors and firm characteristics. Starts, closures, and bankruptcies of businesses directly impact the total number of firms in business.

a. Births, Deaths, and Bankruptcies of Employer Firms

The Small Business Administration (SBA) dynamic business data compiles information on the number of “births” and “deaths” per year for U.S. businesses. The SBA defines firm births as new, original establishments, which were not open in the previous year, and firm deaths as closed original establishments for that year. For the earlier part of the decade (with the exception of 2002), births outnumbered deaths. By 2008, however, the net new number of firms (the number of births minus the number of deaths) was negative: in 2003, net new firms totaled almost 72,000; by 2010, the net loss was close to 60,000 firms.

These figures are reliant on several key factors about when a business is technically “born” or “dead.” When an entrepreneur formally registers may vary from business to business, and the death of a firm also depends on when a business actually winds down its affairs with the proper government entity. EBV Chapter 5, “The Organization and Scale of Private Business” elaborates on these factors, and concludes that it is possible that the data exaggerate the number of firms operating over the course of a calendar year.

See Figure 2-1 on page 33 for firm births, deaths, and bankruptcies for 2000-2010. The only years that estimate a positive net change in the number of employer firms when considering births and deaths was 2006 and 2007. The lowest net change in firms as well as the greatest number of bankruptcies was in 2009, at 162,216 and 60,837, respectively. 2010

Data refers to “openings” as businesses not open in the previous quarter, whereas “births” are the previous year. Similarly, “closings” refers to businesses that closed in the previous quarter, and “deaths” in the year.

Data from Small Business Administration Firm Size Data; U.S. Dynamic Data
data still indicate a negative net change in the number of firms, although the net loss declined to just under 60,000.

FIGURE 2-1. Births and Deaths of Employer Firms (2000-2010)

It is also interesting to look at the proportion of births and deaths by firm size. In 2010, of the 534,000 firm births, over 96% had beginning employment sizes of less than 20. In the same year, only 106 firms with more than 500 employees started, and 150 closed. These trends are fairly consistent throughout the years, as displayed in Table 2-5 on page 34. The amount of large firm births and deaths has decreased over time, but their percentages of total starts and closures did not see significant changes, due to the small proportion of large firms in these categories. This helps to illustrate that many firms are small, and furthermore, the majority of new firms are small, or at least start out small.

53 Note: Data presented is from March-March for one year, and excludes establishments with no first-quarter employment. New firm births are classified by first-quarter employment size. Methodology from 2008 forward defines births/deaths as establishment births/deaths for single-location firms. Previously-displayed data on starts and closures of employer firms exhibit different trends, as the values were taken from a different data source. Figures from alternate source reported earlier in EBV Chapter 5 “The Organization and Scale of Private Business”: http://www.sba.gov/sites/default/files/sbfaq.pdf
TABLE 2-5. Births and Deaths of Employer Firms by Size (2000-2010)

<table>
<thead>
<tr>
<th>Period</th>
<th>Total</th>
<th>Beginning year employment size of firm</th>
<th></th>
<th>Size Class as Percent of</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Firm births</td>
<td></td>
<td>&lt;20</td>
<td>&lt;500</td>
</tr>
<tr>
<td>2009-2010</td>
<td>533,945</td>
<td>514,426</td>
<td>533,839</td>
<td>106</td>
</tr>
<tr>
<td></td>
<td>Firm deaths</td>
<td></td>
<td>572,941</td>
<td>593,197</td>
</tr>
<tr>
<td></td>
<td>Net change</td>
<td></td>
<td>(59,402)</td>
<td>(58,515)</td>
</tr>
<tr>
<td>2008-2009</td>
<td>518,500</td>
<td>497,518</td>
<td>518,382</td>
<td>118</td>
</tr>
<tr>
<td></td>
<td>Firm deaths</td>
<td></td>
<td>655,858</td>
<td>680,516</td>
</tr>
<tr>
<td></td>
<td>Net change</td>
<td></td>
<td>(162,216)</td>
<td>(158,340)</td>
</tr>
<tr>
<td>2007-2008</td>
<td>597,074</td>
<td>570,251</td>
<td>596,483</td>
<td>231</td>
</tr>
<tr>
<td></td>
<td>Firm deaths</td>
<td></td>
<td>613,794</td>
<td>641,158</td>
</tr>
<tr>
<td></td>
<td>Net change</td>
<td></td>
<td>(44,326)</td>
<td>(43,543)</td>
</tr>
<tr>
<td>2006-2007</td>
<td>668,395</td>
<td>639,110</td>
<td>668,177</td>
<td>218</td>
</tr>
<tr>
<td></td>
<td>Firm deaths</td>
<td></td>
<td>564,345</td>
<td>592,148</td>
</tr>
<tr>
<td></td>
<td>Net change</td>
<td></td>
<td>75,985</td>
<td>74,765</td>
</tr>
<tr>
<td>2005-2006</td>
<td>670,058</td>
<td>640,710</td>
<td>669,841</td>
<td>217</td>
</tr>
<tr>
<td></td>
<td>Firm deaths</td>
<td></td>
<td>573,302</td>
<td>599,078</td>
</tr>
<tr>
<td></td>
<td>Net change</td>
<td></td>
<td>76,725</td>
<td>67,408</td>
</tr>
<tr>
<td>2004-2005</td>
<td>644,122</td>
<td>616,019</td>
<td>643,850</td>
<td>272</td>
</tr>
<tr>
<td></td>
<td>Firm deaths</td>
<td></td>
<td>539,061</td>
<td>565,482</td>
</tr>
<tr>
<td></td>
<td>Net change</td>
<td></td>
<td>78,377</td>
<td>76,958</td>
</tr>
<tr>
<td>2003-2004</td>
<td>628,917</td>
<td>601,927</td>
<td>628,655</td>
<td>262</td>
</tr>
<tr>
<td></td>
<td>Firm deaths</td>
<td></td>
<td>515,031</td>
<td>540,746</td>
</tr>
<tr>
<td></td>
<td>Net change</td>
<td></td>
<td>87,870</td>
<td>86,896</td>
</tr>
<tr>
<td>2002-2003</td>
<td>612,296</td>
<td>585,552</td>
<td>611,976</td>
<td>320</td>
</tr>
<tr>
<td></td>
<td>Firm deaths</td>
<td></td>
<td>514,565</td>
<td>540,328</td>
</tr>
<tr>
<td></td>
<td>Net change</td>
<td></td>
<td>71,638</td>
<td>70,987</td>
</tr>
<tr>
<td>2001-2002</td>
<td>569,750</td>
<td>541,516</td>
<td>568,280</td>
<td>1,470</td>
</tr>
<tr>
<td></td>
<td>Firm deaths</td>
<td></td>
<td>557,133</td>
<td>586,535</td>
</tr>
<tr>
<td></td>
<td>Net change</td>
<td></td>
<td>(17,140)</td>
<td>(15,617)</td>
</tr>
<tr>
<td>2000-2001</td>
<td>585,140</td>
<td>558,037</td>
<td>584,837</td>
<td>303</td>
</tr>
<tr>
<td></td>
<td>Firm deaths</td>
<td></td>
<td>523,960</td>
<td>552,839</td>
</tr>
<tr>
<td></td>
<td>Net change</td>
<td></td>
<td>31,149</td>
<td>34,077</td>
</tr>
</tbody>
</table>


b. Establishment Entry and Job Creation

Data presented by the Business Dynamic Statistics (BDS) display a great deal of information pertaining to firm and establishment entry, exit, job creation, and job death by firm size. By definition, an establishment is a fixed location at which economic activity occurs, and a firm may have one or many establishments, compiled based on an aggregation of all
establishments under common ownership by a corporate parent. Therefore, in some cases, an establishment may also be a firm, but the number of establishments outnumbers firms. In this section, the term “businesses” refers to establishments.

The entry rate for any given year, as reported by BDS, is determined by the number of establishments that enter in a year divided by the number of firms that existed at the beginning of that year. An establishment entry is an establishment with positive employment in the current year that has no employment in the previous year. BDS also reports these by establishment age. In aggregate, the entry rate for new businesses has declined from the mid-2000’s, when it was fairly steady around 12%. Between 2008 and 2010, the entry rate decreased to about 9-10%; 2009 marks the lowest entry rate (9%) since 1977, the first year in which data were collected.

Entry and job-creation rates by all establishments followed similar growth patterns between 2000 and 2010, with job-creation rates being steadily higher throughout the entire decade. Job creation is defined as the sum of all employment gains from new or expanding establishments. In 2000, the job-creation rate was as high as 17.4% for firms of all ages and sizes, even though net job creation that year was only 3.2%. In comparison, the job-creation rate in 2010 was lower, at 12.4%, with a net job-creation rate of -1.6%. The net loss of jobs in 2009 is consistent with the impacts felt by the recession.

Despite this, however, the job-creation rate has been consistently higher than establishment entry rates, though they have followed similar patterns over the past decade. Figure 2-2 on page 36 shows establishment entry and job-creation rates between 2000 and 2010. Establishment entry and job creation tend to increase and decrease over similar time periods, but job-creation rate is consistently higher than establishment entry rate.

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54 BDS data provided by the U.S. Census Bureau. More detailed definitions can be found here: http://www.census.gov/ces/dataproducts/bds/overview.html
The job-creation rate above, does not take into consideration job destruction (losses) that occur, solely new jobs that are created by employer firms. Job destruction is the sum of employment losses from firms contracting or shutting down. If job creation exceeds job losses, then net job creation is positive; conversely, if job losses in a given year exceed job creation, then there are fewer jobs in the economy.

The net job-creation rate takes these factors into account. So while the job-creation rate will always be positive (as long as there are new hires), the net job-creation rate may be negative (if the loss of jobs is greater than jobs created).

Table 2-6 on page 37 shows the job-creation rate as well as net job-creation rate by all employer firms. Until recently, the net job-creation rate has been mostly positive over the past decade. Net creation began to decline in 2008, and was negative in 2009 and 2010. This negative net job-creation rate also corresponds with the lowest percentage of job creation in the past decade.
Similarly, the numbers presented for establishment entry rate do not take into account establishments that have exited the industry. Establishment exits are establishments that report zero employment for the year, when there was some employment in the previous year. Similar to net job creation, net establishment entry rates decline significantly when taking establishment exits into account. 2009 did have the smallest value for entry rates and net entry rates, with both rates being the lowest for the decade (9.1% and -2.9%, respectively). Table 2-7 shows establishment entries and exits for the decade, as well as entry and net entry rates. Similar to net new jobs, the net number of new establishments was the smallest in 2009, in the midst of the most recent recession.

TABLE 2-6. Job-Creation Rate and Net Job-Creation Rate (2000-2010)

<table>
<thead>
<tr>
<th>Year</th>
<th>Employment</th>
<th>Job Creation</th>
<th>Job Creation Rate</th>
<th>Job Destruction</th>
<th>Net New Jobs</th>
<th>Net Job Creation Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>112,624,578</td>
<td>19,312,020</td>
<td>17.40%</td>
<td>15,722,124</td>
<td>3,589,896</td>
<td>3.20%</td>
</tr>
<tr>
<td>2001</td>
<td>114,349,928</td>
<td>18,452,856</td>
<td>16.20%</td>
<td>17,619,886</td>
<td>832,970</td>
<td>0.70%</td>
</tr>
<tr>
<td>2002</td>
<td>112,123,658</td>
<td>19,120,170</td>
<td>16.90%</td>
<td>20,880,990</td>
<td>-2,750,820</td>
<td>-1.60%</td>
</tr>
<tr>
<td>2003</td>
<td>112,720,031</td>
<td>18,267,626</td>
<td>16.30%</td>
<td>16,514,168</td>
<td>1,753,458</td>
<td>1.49%</td>
</tr>
<tr>
<td>2004</td>
<td>114,002,476</td>
<td>17,740,729</td>
<td>15.70%</td>
<td>15,538,964</td>
<td>2,161,765</td>
<td>1.90%</td>
</tr>
<tr>
<td>2005</td>
<td>115,520,908</td>
<td>17,880,684</td>
<td>15.60%</td>
<td>15,572,643</td>
<td>2,168,041</td>
<td>1.90%</td>
</tr>
<tr>
<td>2006</td>
<td>118,921,120</td>
<td>19,138,681</td>
<td>16.40%</td>
<td>15,131,184</td>
<td>4,007,497</td>
<td>3.50%</td>
</tr>
<tr>
<td>2007</td>
<td>119,913,219</td>
<td>19,425,583</td>
<td>16.30%</td>
<td>18,061,881</td>
<td>1,363,702</td>
<td>1.10%</td>
</tr>
<tr>
<td>2008</td>
<td>120,083,048</td>
<td>17,419,510</td>
<td>14.60%</td>
<td>16,556,064</td>
<td>863,446</td>
<td>0.80%</td>
</tr>
<tr>
<td>2009</td>
<td>113,900,772</td>
<td>13,797,846</td>
<td>11.80%</td>
<td>19,495,945</td>
<td>-5,698,099</td>
<td>-4.90%</td>
</tr>
<tr>
<td>2010</td>
<td>111,175,010</td>
<td>13,875,971</td>
<td>12.40%</td>
<td>15,684,489</td>
<td>-1,808,518</td>
<td>-1.60%</td>
</tr>
</tbody>
</table>

Source: U.S. Census Bureau, Business Dynamic Statistics
Analysis: Anderson Economic Group, LLC

TABLE 2-7. Establishment Entry Rate and Net Entry Rate (2000-2010)

<table>
<thead>
<tr>
<th>Year</th>
<th>Establishments</th>
<th>Establishment Entry</th>
<th>Establishment Entry Rate</th>
<th>Establishment Exits</th>
<th>Net New Establishments</th>
<th>Net Establishment Entry Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>6,219,280</td>
<td>706,598</td>
<td>11.36%</td>
<td>662,727</td>
<td>43,871</td>
<td>0.71%</td>
</tr>
<tr>
<td>2001</td>
<td>6,348,830</td>
<td>729,529</td>
<td>11.49%</td>
<td>691,775</td>
<td>37,754</td>
<td>0.59%</td>
</tr>
<tr>
<td>2002</td>
<td>6,399,351</td>
<td>817,867</td>
<td>12.78%</td>
<td>757,824</td>
<td>60,043</td>
<td>0.94%</td>
</tr>
<tr>
<td>2003</td>
<td>6,460,594</td>
<td>780,414</td>
<td>12.08%</td>
<td>608,946</td>
<td>171,468</td>
<td>2.65%</td>
</tr>
<tr>
<td>2004</td>
<td>6,542,356</td>
<td>770,133</td>
<td>11.77%</td>
<td>624,933</td>
<td>145,200</td>
<td>2.22%</td>
</tr>
<tr>
<td>2005</td>
<td>6,679,753</td>
<td>811,533</td>
<td>12.15%</td>
<td>621,169</td>
<td>190,304</td>
<td>2.85%</td>
</tr>
<tr>
<td>2006</td>
<td>6,781,915</td>
<td>831,706</td>
<td>12.26%</td>
<td>679,159</td>
<td>152,547</td>
<td>2.25%</td>
</tr>
<tr>
<td>2007</td>
<td>6,888,393</td>
<td>819,510</td>
<td>11.90%</td>
<td>713,598</td>
<td>105,912</td>
<td>1.54%</td>
</tr>
<tr>
<td>2008</td>
<td>6,862,476</td>
<td>713,069</td>
<td>10.39%</td>
<td>704,812</td>
<td>8,257</td>
<td>0.12%</td>
</tr>
<tr>
<td>2009</td>
<td>6,678,469</td>
<td>610,082</td>
<td>9.14%</td>
<td>805,721</td>
<td>-195,639</td>
<td>-2.93%</td>
</tr>
<tr>
<td>2010</td>
<td>6,619,139</td>
<td>668,861</td>
<td>10.10%</td>
<td>690,504</td>
<td>-21,643</td>
<td>-0.33%</td>
</tr>
</tbody>
</table>

Source: U.S. Census Bureau, Business Dynamic Statistics
Analysis: Anderson Economic Group, LLC
c. Survival and Employment by Entity Age

Entering the market and attracting employees are only the first steps for business entities. Firms and establishments must also survive within their industries and retain employment. The survival rate of a business is dependent on many factors.

**Survival and Employment by Age of Establishment**

The age of an establishment itself, and not the firm that owns it, is associated with many different characteristics, including entry and exits. In general, trends indicate that establishments that are older have a higher survival rate, as defined by the amount of businesses that have not exited for a given year: the annual survival rate for establishments aged 26 or older has been around 95% since 2002. In contrast, establishments that are between zero and five years is much lower, ranging between 83% and 88% in the past decade. Figure 2-3 below details establishment-survival-rate trends over the past decade.

![FIGURE 2-3. Establishment Survival Rate by Age (2000-2010)](image)

*Note: Data began being collected in 1977, therefore data on establishments aged 26+ is only available from 2003 onwards.*

*Source: U.S. Census Bureau, Business Dynamic Statistics*

*Analysis: Anderson Economic Group, LLC*

The trends for establishments of all ages follow similar patterns in terms of periods of economic growth and decline. For all businesses, the average survival rate has been approximately 90% for the past decade. One particular note of interest is that younger businesses seem to have been the

55 Survival rates are shown based on age of the establishment itself, and not the firm by which it is owned.
hardest-hit by the economic recession in 2009. The difference in percentage points for survival between 2008 and 2009 perfectly corresponds to establishment age groups, with the youngest experiencing the greatest decline at 2.8 percentage points, and the oldest experiencing the least decline over the previous year, at 0.7 percentage points.

Another interesting variable to consider is the proportion of national employment contributed to by businesses of varying age groups. Despite having lower survival rates, younger businesses (aged 0-5) have the largest employment contribution of any age group. In the past several years, younger establishments have contributed slightly less to national employment (with the exception of 2010), although they have consistently provided over a quarter of national employment. Figure 2-4 on page 39 below displays the proportion of national employment by each age group since 2000, showing that younger businesses consistently have contributed the largest proportion of employment since 2000.

**FIGURE 2-4. Proportion of National Employment by Establishment Age (2000-2010)**

Survival and Employment by Age of Firm

Firms, as stated earlier, may have one or many establishments. The following subsection presents data on establishments based on the age of the firm that owns them. Looking at establishments by firm age presents the data from a different view. In general, similar trends hold as the ones...
that emerge when comparing establishment survival rate by age: establishments owned by older firms have a higher survival rate.

In particular, however, establishments that are associated with younger firms have a lower survival rate than when looking at young establishments, regardless of firm age; survival rate for establishments for firms aged zero to five was 82.6% in 2010, the lowest of all of the age groups. For comparison, recent survival rate for establishments for firms aged at least 26 is almost 95%. Figure 2-5 indicates survival rates for establishments based on firm ages, highlighting the tendency for older firms to have a higher proportion of establishment survival.

FIGURE 2-5. Establishment Survival Rate by Firm Age (2000-2010)

Analyzing employment data for establishments based on firm age can also present interesting information. Establishments for older firms (particularly with an unknown birth year prior to 1976), represent nearly half of national employment. This proportion is far larger than the amount of employment represented by older establishments. The explanation for this variation is that older firms may be opening new establishments throughout the years; taking firm age into account indicates that older businesses have a higher share of the workforce. Figure 2-6 shows the contribution to employment by firms of different ages, highlighting the significance of older firms.
There are many different ways to analyze data by firm characteristics. We can break down firms by size, ownership, corporation type, and age, among many other distinguishing characteristics. Analyzing the available data helps to highlight the importance of smaller, privately-held firms, that greatly contribute to the nation’s economy as a whole. There is also an inherent degree of correlation between ownership, firm type, and size. The combination of these factors provides meaningful insight into which firms and establishments contribute to employment, earnings, and other national economic factors. With the recent explosion of corporations involved in industries such as technology, it will be interesting to observe the contributions of younger firms to national employment in the years to come.
Chapter 3

Economic Theories of Value

by Patrick L. Anderson

An Introduction

The neo-classical model of economics suggests that without interference the market will efficiently allocate resources, goods, and services. Furthermore, this basic theory is the basis for the modern concept of value: prices are set when supply equals demand. In this chapter, we discuss differing notions of neoclassical economics, as well as walk the reader through the mathematics of this model.

Section 3.1. Distinguishing Notions of Neoclassical Economics

In EBV Chapter 8 “Value in Neoclassical Economics,” the author concludes that the neoclassical theory is not a sound basis for determining the market value of a privately-held firm because no such firm actually fits into the neoclassical model. There is only one pervasive example in business of true one-period profit maximization that is easily identified. This is behavior we prohibit by law: criminal business activity.

Many persons could, with relatively little risk, increase the money they have by turning to crime. Why don’t they? The major reason (we would hope) is that they find it immoral to do so. Another reason is that the risks and penalties are too great when compared to the payout. If we boil it down to just risk and reward in a short time frame, crime does pay. If what you truly care about is “profit maximization,” crime beats honest work. Even criminals know this.
A very clear explanation of this dynamic was provided by former street criminal, and current rap music artist, “Ice T,” in an interview reprinted in the *Guardian* (UK):\(^{56}\)

Ice T: When I got to high school, I started seeing people with things that were beyond their means. I saw guys in the 12th grade that got cars and you're looking at everybody asking yourself what was really going on. Then my boys started to sell weed. They would buy pounds of home-grown weed, then sell joints for a dollar. Then my other friend, his sister used to steal the lunch ticket roll from the school and sell the tickets. She was making a lot of money. She put us to work. She said, “You guys sell them and I'll give you half the money.” I was like, “This little broad is running an organised crime ring.”

When you get into that you start to hang around with people who take pride in beating the system. That's the scary part. Everyone around you says, “If you ain't a hustler then you're not cool.” You believe stolen money is good, and working money is not.

Why should entrepreneurs be considered any more or less insightful, moral, or enterprising than potential criminals? They shouldn’t; it is the “profit maximization” model that needs to change. One-period profit maximization in business is a road to disaster, and a road that entrepreneurs by and large do not travel. However, there are several aspects of neoclassical economics that are useful in framing our thoughts about value, e.g. price is determined by supply and demand. Below we describe the math behind the neoclassical model, which serves as the foundation of microeconomics; in particular, we focus on consumer preferences and utility, time preferences, and risk aversion.

\section*{3.2. Mathematics of the Neoclassical Economics Model}

The neoclassical economics model was formalized in a concise manner by the French-born economist Gérard Debreu (1921-2004), who emigrated to the United States and won the 1983 Nobel Memorial Prize in Economics.\(^{57}\) His 1959 monograph *The Theory of Value: An axiomatic analysis of economic equilibrium* is one of the most influential books in mathematical economics. We follow here the presentation of that text.

\begin{thebibliography}{1}
\end{thebibliography}
a. Commodities, Prices, Value

**Commodities**

Commodities include goods or services, which are defined by the physical structure, time of delivery, and place of delivery. There is a defined, integer number of commodities that can be indexed $1...n$.

“Consumption” in economics includes purchases or use of all the following commodities:

1. Both goods and services of all types, including: staple, fancy, and convenience foods; basic housing, luxury housing, neighborhood amenities, view, and location; necessary household expenses, clothing, entertainment, travel, and luxuries; education; and participation in status-enhancing activities.
2. The avoidance of certain undesirable commodities or experiences; this allows the removal of garbage and environmental clean-up costs to be treated in a normal fashion.
3. Charitable contributions, bequests, and gifts to others; this set recognizes that humans actually consider these important and often act in a generous manner towards others.
4. The costs of insurance and expenditures on other items that reduce risk or enhance a consumer’s feeling of satisfaction.

A group of commodities is known as a commodity bundle.

**Prices**

Most commodities have a general range of associated prices for a set quantity. The price of a commodity is a real number indicating payment made for the delivery of a commodity at a specified time. Prices in this manner include wages, rent, fares, fees, salaries, royalties, and prices on consumer goods.

The price is typically positive (for “scarce” commodities); however, it could be zero (“null” or “free”), or even negative. Depending on the

---

57 Some scholars distinguish the “Neo-walrasian” general equilibrium theory of Debreu from other general equilibrium theories within neoclassical economics. However, the ubiquity of the Debreu formulation appears to settle the question of the foundational beliefs of neoclassical economics.
convention, negative prices could indicate a noxious commodity for which a person would pay for removal.

The price system can be represented by a vector of prices of individual commodities. For a system of \( n \) commodities, this would be an \( n \)-tuple of prices \( p \). The value of an action \( a \) is the inner product of the price vector and the vector of individual actions:

\[
\text{Value of action in microeconomics (EQ 3-1)}
\]

\[
\text{value of action} = \sum_{k=1}^{n} p_k a_k = p \cdot a;
\]

where:

\[
p = (p_1, \ldots, p_k, \ldots, p_n), \text{ a price vector } n\text{-tuple;}
\]

\[
a = (a_1, \ldots, a_k, \ldots, a_n), \text{ an action vector } n\text{-tuple.}
\]

Common actions pertaining to commodities include buying and selling. If an action is simply to buy one unit of the first-indexed commodity, the value of the purchase would be \( a_1 p_1 = 1 p_1 \).

The model is normalized by choosing one commodity to be the basis for a specific price index. This commodity is often termed the \textit{numéraire}. In many settings, simply choosing a specific currency unit (dollars, euros, lira, Monopoly money, etc.) accomplishes this. In others, the price index must be specified carefully, and allowance made for changes in the price level over time (inflation or deflation).

Similarly, a set of locations is taken into consideration, in which prices may vary at the same time for different commodities. When these locations are partitioned into nations, each with its own currency, \textit{exchange rates} allow for the comparison of one to the other at a specific time. In most micro economic settings, this is ignored.

**Discounting across time**

While commodities are associated with prices at a specific time, the economy develops across a continuum of time. Additionally the prices of commodities (which, it is useful to recall here, include services, goods, wages, rents, and royalties, etc.) vary across time.
One way to incorporate time into prices is to use a discount rate, which is defined as the fraction that converts the price paid at the current time for a commodity to be purchased at a later time:

\[ p_{t-1} = \beta p_t; \]

where:

- \( p_{t-1} \) = price paid for commodity at time \( t - 1 \).
- \( t = (0, ..., T) \) time index;
- \( 0 < \beta < 1 \) gross discount factor.

The convention here on the form of the discount rate is different from that commonly used in accounting and business reporting in the United States, and is discussed in “Conventions for interest, discounting, and gross returns” on page 47.

**Saving and Investing**

The ability to purchase commodities implies a consumer having income or endowment for a specific time period; however, a consumer may choose to not consume a portion of their income or other endowment in one period. The consumer is considered to have saved the balance remaining, and this balance is called savings. Note that there is no economic difference at this level of abstraction between savings and investment on the part of the consumer; they are identically characterized as the amount of consumption that could be paid out of current income, but is instead deferred.

The consumer’s savings in one period may grow to a larger amount in subsequent periods.\(^{58}\) A common application of this would be the sum of interest payments received from a principal deposited in a savings account, or the amount received at the maturity of a pure discount bond (compared to the original purchase price of the bond). If we illustrate

\[ \alpha = B_t / B_s \]

---

\(^{58}\) In Debreau’s original description of the neoclassical model, the rate of accumulation \( \alpha \) was defined as the fraction that related the amount received at a later date for the amount saved at the earlier date:
both of these using the convention of the value of a bond at time period \( t \) as \( B_t \) (and consider the bank deposit in the same manner as a short-term discount bond), we have the following relations:

\[
B_1 = B_0 (1 + r) = B_0 R
\]

By convention, the net interest rate \( r \) is assumed to be a positive fraction less than one, although there are empirical exceptions to this. We will recognize Debreu’s concept of the rate of accumulation as the contemporary \textit{gross rate of return}, which is denoted \( R=(1+r) \).

\textbf{b. Consumers; Utility}

Consumer tastes can be modeled by their \textit{utility functions}, which map preferences about commodities to real numbers, which can then be ordinally ranked. Utility functions have certain properties that make them well-behaved. We discuss utility functions in Section A.29. “Utility Functions” in the Appendix.\(^{59}\)

Graphically representing the aggregation of consumer demands given a range of possible prices, depicts the \textit{demand curve}. Note that the classic downward-sloping demand curve is a graphical illustration of the neoclassical theory of consumer demand, using the convention popularized by Alfred Marshall, incorporating common assumptions about consumers and producers.

\textbf{c. Conventions for interest, discounting, and gross returns}

In presenting the neoclassical foundation outlined by Debreu, as well as in other economic derivations we present in this book, we will use certain conventions of the expression of discount rates that are different than the form often found in finance and accounting books. We summarize them here.

We use \( d \) to represent a per-period fractional reduction in the amount one would be willing to pay today in return for a future payment. We call this a \textit{net discount factor}. We use the symbol \textit{beta} to denote a \textit{gross
**discount rate** that directly converts the price paid tomorrow for the price paid today, as in equation (EQ 3-2). The relationship between the gross and net rates is illustrated in the following equation.

\[
\beta = 1/(1 + d)
\]  

*(EQ 3-4)*

We use the symbol \( r \) to denote the fractional return per period on lending (or fractional payments on borrowings), as in equation (EQ 3-3). This is the same usage as the most common concept of “interest rates” for consumer lending or deposit accounts in the United States. We use the symbol \( R \) to denote the corresponding gross rate of return. Depending on the type of security, we may modify either symbol to denote a specific interest rate, such as the “risk-free rate.”

If, as occurs often in theory but rarely in practice, the interest rate is the same as the discount rate used by both parties to a contract, the following relationship exists:

\[
\beta = 1/(1 + d)
\]

*(EQ 3-5)*

Thus, for an interest rate of \( r = d = 10\% \), the discount rate \( \beta = 1/(1 + d) = 1/(1 + r) \); \( \beta = R^{-1} \).

**d. Producers in the Neoclassical Model**

Consumers only account for one part of supply and demand models; producers fall on the other side of the equation. The following is an outline of the neoclassical model of the firm. This model uses calculus and optimization, the basics of which are outlined in Section A.26. “Optimization Using Differential Calculus” in the Appendix.
The actions of the firm include purchasing supplies, hiring workers, setting production plans, purchasing advertising, and selling the goods and services it produces. These actions can be summarized in two sets of equations. The first identifies the actions $a_1, \ldots, a_n$ as arguments to functions that determine the revenue and costs of the firm:

\begin{align*}
\text{Revenue and cost functions} \quad (\text{EQ 3-6}) \\
R &= R(a_1, \ldots, a_n) \\
C &= C(a_1, \ldots, a_n).
\end{align*}

The second establishes the objective of the firm as the maximization of profit, which is the difference between revenue and costs:

\begin{align*}
\text{Neoclassical objective of the firm: profit maximization} \quad (\text{EQ 3-7}) \\
\max_{a_i} \pi &= R(a_1, \ldots, a_n) - C(a_1, \ldots, a_n); \\
a_i &> 0; i = 1, \ldots, n.
\end{align*}

The firm seeks to maximize profits by changing its actions until it finds $a^*$, the set of actions that collectively maximizes profits. (Recall that this is a one-period model.) Using calculus (and the assumptions of the neoclassical model), we know the maximum of a function of several variables is a stationary point. Here the partial differential of revenue with respect to each action at least exceeds the partial differential of costs with respect

---

60 This presentation is based on Debreu (1959, chapter 3). I have also referred to the first edition of Hal Varian’s *Microeconomic Analysis* (1978, chapter 1). There are many, many other texts that repeat these same concepts and often use the same equations, including later editions of Varian’s *Microeconomic Analysis*. I rely on them because Debreu created the modern synthesis and Varian authored one of the seminal graduate textbooks used in American universities.

61 The stationary point is where the partial differentials equal zero. In this case, simplifying the functions makes the “first derivative equals zero” clearer. Note that the derivative of the profit function would be zero when marginal revenue equals marginal cost:

\begin{align*}
\pi &= R(a) - C(a); \\
\frac{d\pi}{dx} &= R'(a) - C'(a).
\end{align*}
to the same action, and where the action is positive, the differentials for cost equal those of revenue.

Calculus of Profit Maximization (EQ 3-8)

\[
\frac{\partial R(a^*)}{\partial a_i} - \frac{\partial C(a^*)}{\partial a_i} \leq 0; \\
\frac{\partial R(a^*)}{\partial a_i} = \frac{\partial C(a^*)}{\partial a_i} \text{ if } a_i > 0.
\]

This is the famous “marginal revenue equals marginal cost” equation.\textsuperscript{62} Varian (1978, chapter 1) summarizes this as follows:

The basic assumption of most economic analysis is that the firm acts so as to maximize its profits; that is, it chooses actions \((a_1, a_n)\) as to maximize \(R(a_1, a_n) - C(a_1, a_n)\).
Chapter 4

Finance Theories of Value

by Patrick L. Anderson

An Introduction

In this chapter we offer additional perspectives on finance theories that were discussed in EBV Chapters 10 through 12. These include arbitrage-free pricing in complete markets, portfolio pricing methods, and real options. Below we discuss alternatives to the complete market pricing formulas shown in EBV Chapter 10. We then discuss other derivations of portfolio pricing methods, including CAPM.

SECTION 4.1. Alternate Presentations of Complete-Markets Pricing Formulas

Recall the basic pricing equations from EBV Chapter 10 “Arbitrage-Free Pricing in Complete Markets.” Below we present other pricing formulas which describe how consumer-investors choose to save in a multiple-period setting of the modern recursive model.

Alternate Derivation 1: State Pricing

Within a complete market, the risk-neutral pricing formula, and its alternate representation as the state price density pricing formula, imply a positive, linear operator on payoffs. By “positive, linear operator,” we mean a positive vector of numbers that, multiplied by the payoffs possible in the various future states of nature, provide the current price of the asset.
Positive linear pricing operator in complete markets

\[ V(z) = qz \]

Now assume we have an asset \( e \) that has a payoff of \( 1 \) in every state at the end of the single time period. This is a risk-free asset; a pure-discount bond.\(^{63}\) The value of the pure-discount bond must be the gross discount rate, multiplied by one (the payoff in every state):

\[ \text{Risk-free Rate and Linear Pricing Operator} \quad (\text{EQ} \ 4-2) \]

\[
\begin{align*}
 e_i &= e = 1, \text{ all } i = 1...m; \\
 \frac{1}{R_f} &= \frac{1}{1 + r_f} = V(e); \\
 r_f &= \frac{1}{V(e)} - 1
\end{align*}
\]

In fact, this relationship defines a special rate of return: the “risk-free” rate. We will use the notation \( R_f \) to denote the gross risk-free rate, and \( r_f \) the net risk-free rate, throughout this text.

Now return to equation (EQ 4-1) shown above. If the payoff vector \( z \) is the payoffs of the default-free bond \( e \), we can now infer that the sum of all the linear pricing weights \( q_i \) must be one times the gross discount rate for a risk-free asset:

\[ \text{Risk-free rate and probability weights} \quad (\text{EQ} \ 4-3) \]

\[ V(e) = \frac{1}{1 + r_f} = qe = \sum_i q_i \]

---

\(^{63}\) A pure discount bond is one that has a lump-sum payoff at maturity. By contrast, some bonds pay “coupon” interest during their term, or have periodic principal payments. The price of a discount bond is a direct indication of the gross discount rate.

Thus, if the risk-free interest rate was \( r=5\% \) per annum, then the price at the beginning of the period of a $100 pure-discount bond yielding that net rate of return would be \( \frac{1}{(1+r)/100} = $95. \)

In actual trading situations, part-year periods, commissions, day-count interest, and other factors would also affect the price.
Recall the “State Price Density Formula” found in EBV Chapter 10. That equation expressed the current price as a probability-weighted average of the possible future payoffs, scaled by the state price density. Using that equation and the definitions above, we can isolate the risk-neutral probabilities in terms of the linear pricing operator weights, and summarize the relations among the risk-neutral probabilities, the pricing kernel (or state price density), and the state prices:

\[ \tilde{\pi}_i = \frac{q_i}{\sum_i q_i} \geq 0; \sum_i \tilde{\pi}_i = 1; \]

where:
- \( \tilde{\pi}_i \) = risk-neutral probabilities
- \( \phi_i = \frac{q_i}{\pi_j} = \frac{\left( \frac{1}{1+r} \right)^{\tilde{\pi}_j}}{\pi_j} \) = pricing kernel = SDF
- \( q_i = \frac{\phi_i}{\pi_i} = \left( \frac{1}{1+r} \right)^{\tilde{\pi}_i} \) = state prices

Alternate Derivation 2: Equivalent Martingale Pricing

At the risk of inundating the reader, we will mention yet another interpretation of these equations. This interpretation emphasizes the difference between the actual probabilities and the “risk-neutral” probabilities, by calling the relationship between the two a “change of measure.” The “measure” here is the determination of a number under measure theory, and important basis for statistics.64

Note that these formulas give the price today as the weighted average of the discounted payoffs tomorrow. A series of numbers generated by a stochastic process, in which the expected value of the next number is the current number, is known as a martingale.65 Thus, the discounted price process here is a martingale, because the expected value of the discounted end-period payoff is the beginning-period price.

64 See Section A.19. “Random Variables, Measure Theory” in the Appendix.
65 Stochastic processes are outlined in Section A.20. “Stochastic Processes” in the Appendix.
The state price density is defined using the same concepts as the risk-neutral probabilities; the two formulas are equivalent expressions of the same underlying theory. The “equivalent martingale pricing” interpretation is another variation, which is most relevant when considering the evolution of the state and asset prices over time.

**No Arbitrage Principle**

Another way of formally stating this principle is the following:

1. Define a “stacked” matrix of negative of the current price vector $p$ and the tableau of possible payoffs $G$.

2. Multiply this by the portfolio weight vector $\eta$ to arrive at the net cost, at this-period’s prices, of payoffs in the next period. The matrix algebra involved automatically subtracts the prices paid in the current period (the cost of the portfolio) from the payoff in all states in the next period.66

**Example of Arbitrage (EQ 4-5)**

$$A = \begin{bmatrix} -p \\ G \end{bmatrix}; \quad A\eta > 0$$

3. Such a certain payoff would be indicated by a positive product of the portfolio weight vector and the stacked matrix. By “positive,” we mean all elements are at least zero, and at least one element is positive.

4. The No Arbitrage assumption is a statement that the set of portfolios that are an arbitrage is the null set.

**No Arbitrage Statement (EQ 4-6)**

$$NA \iff \{\eta \mid A\eta > 0\} = \emptyset$$

---

66 If this isn’t obvious to you, don’t be intimidated; the author needed to write a computer program and run it several times to convince himself!
Fundamental Pricing Equation for Complete Markets: Multiple-Period Assets

Under the assumptions of complete markets and the validity of the Fundamental Theorem of Asset Pricing, we can derive a pricing equation for assets providing payoffs over time.\(^67\)

\[
p_0 = \hat{E}[\int_0^T e^{r(s)ds} z_T];
\]

where:

- \(p_0\) = price at time zero;
- \(z_T\) = random payoff at time \(t = T\);
- \(\hat{E}[\bullet]\) = expectation under risk-neutral measure; and
- \(r(s)\) = instantaneous risk-free interest rate

This form of the equation (in continuous time, using integral calculus) is more opaque than the discrete-time equivalent.\(^68\) However, it also illuminates why the “twisted” risk-neutral probabilities are also known as “equivalent martingale measures:” the discounted (using the risk-free rate) expected future value is a martingale.

An alternate form of the equation, which is extended to multiple-payoff securities (where the complete-markets assumption extends to multiple time periods),\(^69\) is the following:

---

67 This equation can be stated “without much imagination,” according to Ross (2005, chapter 1); it apparently needs at least a little imagination in texts such as Fusai & Roncoroni (2008, chapter 1) or Singleton (2006, chapter 1), which note the extensive restrictions implied by the equation.

The equation is shown in these three texts with only one payoff (e.g. at \(t = T\)); see below for the extension to multiple-period securities.

68 For an explanation of the discounting term in this equation, see EBV Chapter 12 “Real Options and Expanded Net Present Value.”

69 Given the law of one price in complete markets, the formula is also useful for securities with multiple payoffs across time, under the assumption that complete markets extend across these time periods. If this is not the case, or if the payoffs depend on each other in a manner that is not captured by the securities that replicate the individual payoffs, the equation may not hold.
Alternate Fundamental Pricing Equation in Complete Markets (EQ 4-8)

For single payoff $z_T$:

$$p(z_{T,0}) = \tilde{E}[\int_{t=0}^{T} e^{-r_t} z_t dt]$$

For series of independent payoffs $\{z_t\}$:

$$p(z_{T,0}) = \tilde{E}(z_T)[\int_{t=0}^{T} e^{-r_t} dt] + \tilde{E}(z_{T_2})[\int_{t=0}^{T_2} e^{-r_t} dt] + ...$$

$$= \tilde{E}[\int_{t=0}^{T} e^{-r_t} z_t dt]$$

The same result can be presented in alternate notation, again confined to the payoff space of the basic securities in a no-arbitrage, complete markets model, showing that the prices of all securities are the present value of their future payoffs, where the “present value” is taken with respect to the risk neutral measure.\(^{70}\)

**SECTION 4.2. Alternate Derivations and Portfolio Models**

**Derivations on Return-Beta Models\(^{71}\)**

Many models used in Finance texts are actually special cases of the Basic Pricing Equation $p_t = E(m_t + I x_t + I)$, and can be derived from it. Using the recursive form of the consumer’s savings problem, allows for an explicit consideration of consumer “wealth” as well as “income.”

In these derivations, it is worth noting the following:

- $E(m)$ must be positive, if there is an absence of arbitrage. This rules out dividing by zero in the equations. Furthermore, an expected value of the pricing kernel near zero implies a near-infinite discount rate on the risk-free asset—an extreme and rarely-observed behavior among actual investors.

\(^{70}\) See Magill & Quinzii (1998, section 9, definition 9.6).

\(^{71}\) This section follows the presentation of Cochrane (2001, section 6.1-6.2).
The definition of the expectation of a product of two random variables provides a crucial assist in this derivation: 
$$E(mR) = E(m)E(R) + \text{cov}(m,R).$$

The $\lambda$ term is not another parameter to be estimated; it is a variable expressing market information. Thus, the equation expresses the expected return on a security as the sum of two factors: an intercept and the product of a market variable and a $\beta$ factor.

Those readers familiar with the more common presentation of “beta” models can confirm that the $\beta$ factor here is a ratio of a covariance to a variance, just as in the classic CAPM model.

**Return-Beta Models**

Derivation of Return-Beta Model (EQ 4-9)

\[ p_i = E(mx_{..}) = E[m(p_{..} + y_{..})] \]

\[ 1 = E \left[ m \frac{(p_{..} + y_{..})}{p_{..}} \right]; \text{dividing by } p_{..} \]

\[ 1 = E \left[ m R \right] = E(m)E(R) + \text{cov}(m, R); \]

using definitions of return and expectation

\[ E(R) = \frac{1}{E(m)} - \frac{\text{cov}(m, R)}{E(m)}, \text{ for security } i \]

Define \( \gamma = \frac{1}{E(m)} \) and multiply right term by \( \frac{\text{var}(m)}{\text{var}(m)} \);

\[ E(R) = \gamma + \beta \cdot \lambda; \]

where:

\[ \beta = \frac{\text{cov}(m, R)}{\text{var}(m)}; \]

\[ \lambda = \frac{-\text{var}(m)}{E(m)} \]

**Linear Factor Models**

Above we show that the basic pricing equation implies a return-beta model. Below we assert that a similar underlying structure exists with
linear factor models of two common types: the “excess returns” models, and the multi-factor models. This result is asserted in the following two theorems; proofs are in Cochrane (2001, section 6.1, 6.2):

- **BPE Equivalence Theorem 1: Excess Returns Model**
  
  Given \( p_t = E(m_{t+1}x_{t+1}) \); where
  
  \[ m_{t+1} = a + b' f_{t+1}, \text{ and} \]
  
  \[ 0 = E(mR^t); R^t = \text{excess returns}; \]
  
  \[ E(R^t) = \beta' \lambda; \lambda \text{ exists}; \text{ and} \]
  
  \[ \beta = \text{regression coefficients on factors } \lambda. \]

- **BPE Equivalence Theorem 2: Multi-factor Models**
  
  Given \( p_t = E(m_{t+1}x_{t+1}) \); where
  
  \[ m_{t+1} = a + b' f_{t+1}, \text{ and} \]
  
  \[ 1 = E(mR^t); R^t = \text{return on security } i; \]
  
  \[ E(R^t) = \gamma + \lambda' \beta_i, \]
  
  \[ \gamma \text{ and } \lambda \text{ exist}; \text{ and} \]
  
  \[ \beta_i = \text{regression coefficients on factors } f_{t+1}. \]

**Quadratic Value from Recursive Quadratic Utility**

How do we make the transition from utility—which is properly based on consumption in one period—to the value that consumers place on wealth? Much, though not all, of that can be done by recasting utility into a recursive formulation. Here, a multi-period problem is recast into a series of 2-period decisions.

In the first equation below, a consumer-investor’s utility in the current period is the expected discounted sum of current and future utility from consumption. In the second equation, a value functional is created by maximizing that utility function across a range of variables under the control of the maximizer, including a stream of consumption decisions and a matching stream of portfolio decisions.
The conditions below the equation summarize other key identities: the time discount factor; the transition equation that produces wealth in the next period from consumption and investment decisions in the current period; the transversality condition that rules out negative wealth; and the return and portfolio weight vectors.72

Recursive Utility and the Value Functional (EQ 4-12)

\[ U_t = E_t \left[ \sum_{j=0}^{\infty} \beta^j u(c_{t+j}) \right]; \]

\[ V(W_t) = \max_{\{c_{t+1}, c_{t+2}, \ldots, w_t, w_{t+1}, w_{t+2}, \ldots\}} (U_t); \]

where:

\[ \beta = \text{discount factor, } 0 < \beta < 1; \]

\[ W_{t+1} = R^w_{t+1} (W_t - c_t), \]

\[ R^w_{t+1} = \sum_{j=1}^{N} w_j R^j_{t+1} = w_{t+1}' R_{t+1}, \]

\[ W_t > W; \ W = \text{a minimum level of wealth} \]

This functional equation is also known as a Bellman equation, and the method is described in Section A.28. “Dynamic Programming” in the Appendix. The recursive construction of the value functional is clearly outlined in the following equation:

72 The total return is shown calculated with both summation and matrix notation. In matrix notation, the inner product of the vector of returns \( R \) on individual securities and the vector of portfolio weights \( w \) is the total return on the invested assets.
Construction of the Recursive Value Functional (EQ 4-13)

\[ V(W_t) = \max_{\{c_t, w_t\}} \left( U_t \right) \]

\[ = \max_{\{c_t, w_t\}} \left\{ E_t \left[ \sum_{j=0}^{\infty} \beta^j u(c_{t+j}) \right] \right\} \]

\[ = \max_{\{c_t, w_t\}} \left\{ u(c_t) + \beta E_t \left[ u(c_{t+1}) + \sum_{j=2}^{\infty} \beta^j u(c_{t+j}) \right] \right\} \]

\[ V(W_t) = \max_{\{c_t, w_t\}} \left\{ u(c_t) + \beta E_t [V(W_{t+1})] \right\} \]

In the first equation, the summation of multiple periods is illustrated as a series of results from specific time periods. In the second, we combine these into the sum of the current period, and the discounted expected value of the next period. For each period, the maximization occurs over consumption and investment decisions, which are summarized in the control variables set \{c, w\}.

**CAPM from the Recursive-Quadratic Utility Model**

Returning now to the derivation of the CAPM within the model, we can derive the CAPM stochastic discount factor:

\[ m_{t+1} = \frac{-\beta \eta (W_{t+1} - W^*)}{u'(c_t)} \]

\[ = \left[ \frac{\beta \eta W^*}{u'(c_t)} \right] + \left[ \frac{\beta \eta W_t - c_t}{u'(c_t)} \right] \tilde{R}_t^{w} \]

\[ m_{t+1} = a_t + b_t \tilde{R}_t^{w} \]

This SDF has an intuitive understanding: an intercept term that is based on the desired level of wealth, a subjective discount factor, and marginal benefits of current consumption; and a “beta” term that is related to the
amount that is invested and the discount factor, along with the marginal benefits of consumption. It is this latter term that is multiplied by the market return in the familiar CAPM equation.

**Classic CAPM: Alternate Derivations**

The EBV author believes the derivation of the CAPM presented above is the most useful for understanding its powerful implications, and the similar limitations. There are other derivations that are based on different fundamental assumptions. These include:

**Two-Period Quadratic Utility Investors**

This derivations builds on three assumptions.

1. First, we assume that investors live for only two periods. That is a much better basis than one-period models, because it acknowledges the common human concern about the future.
2. Second, we assume their entire consumption stream depends on their initial wealth and their investment earnings. This is not a reasonable assumption, as most consumers view their labor earnings (including wages, salary, bonuses, and benefits) as the basis for their living expenses.73
3. The third assumption concerns the form of the utility function for these consumer-investors. In this derivation, we assume a quadratic function, meaning that the mean investment return is the indicator of expected value, and the variance (which is calculated using a quadratic sum of deviations from the mean) is the measure of risk.74 There is ample evidence that some portion of human behavior is described by quadratic preferences.75 However, we must note that such an assumption clearly does not take into account the risk aversion that humans clearly evidence about their own subsistence.

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73 Extending this, we could observe that workers typically view their earnings in retirement years as being fundamentally based on private and public pensions funded largely by contributions and taxes out of labor earnings. Of course, there are many disjunctions between direct labor earnings and future pension benefits, but the principle is strong enough to easily accept the extension of the assumption in this case.

74 For a discussion of the variance, see the Section A.24. “Statistics of Central Tendency and Variation” in the Appendix.

75 See the short section on the “Quadratic Human” in Anderson (2004b).
Note that the second and third assumption isolates consumer behavior about investments after they have their basic subsistence needs satisfied. We assume consumers are endowed with some wealth at the beginning. (To keep the model from falling apart, we must assume enough wealth that consumers can survive for two periods, even if investment returns are relatively “low”). Thus, investment behavior is really about happiness in the second period, because investors enjoy all of the benefits then.

The derivation of the CAPM with these assumptions is straightforward; the presentation by Cochrane (2001, section 9.1) is a good example. Because consumers already have some wealth at the beginning of the two periods, they need only choose how much to consume, and how much of the available investment securities to buy with their remaining wealth. The wealth in the second period becomes their consumption, and by assumption, this consumption is therefore the marginal utility they receive for investment returns. This produces the following budget constraint relating wealth and consumption in the second period (and therefore investment in the first period) to earnings on the portfolio of investments:

Two-Period Quadratic Utility Model  
\[
\begin{align*}
    c_{t+1} &= W_{t+1}, \\
    W_{t+1} &= R_{t+1}^W (W_t - c_t), \\
    R_{t+1}^w &= \sum_{i=1}^{N} w_i R_{t+1}^i; \text{ the "rate of return on wealth," where:} \\
    w_i &= \text{portfolio weights for security } i \in \{1, ..., N\}; \\
    \sum_{i=1}^{N} w_j &= 1.
\end{align*}
\]

Recall that the stochastic discount factor (summarized in Equation 11.1 of EBV Chapter 11) contains a ratio of marginal utility from consumption in the next period to consumption in the current period (the marginal rate of substitution). A quadratic utility function produces linear marginal utility, and substituting the two-period model elements above into the basic pricing equation produces the stochastic discount factor for the quadratic 2-period model:
This, assuming that the wealth portfolio is the available set of securities, is the CAPM.

**Exponential Utility, Normal Distribution of Expected Returns**

Another derivation of the CAPM uses a different set of assumptions:

1. An exponential utility function:\(^{76}\)

\[
E[u(c)] = E\left[-e^{-\alpha c}\right];
\]

where:

- \(c\) = consumption; and  
- \(\alpha\) = coefficient of absolute risk aversion

Here the utility function simplified to a form in which a risk aversion parameter is quite explicit.

2. Investment returns that are normally distributed.

3. One or two-periods; if the number of periods is more than one, all the wealth is consumed in the last period.

The derivation of this model (shown also in Cochrane (2001, section 9.1)) is similar to that of the two-period model. It produces the following expected return formula:

\[m_{t+1} = a_t + b_t R_{t+1}^{w}\]  

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\(^{76}\) Exponential utility should not be confused with logarithmic utility function \(U=\ln(c)\). Indeed, a log utility model can be used to derive the CAPM as well; see Cochrane (2001, section 9.1) for a comparison with other derivations. Cochrane attributes the log-utility CAPM to Rubinstein (1976), which contains a CAPM-like equation using very restrictive assumptions and log-utility, although not an explicit CAPM.
Exponential Utility, Normal Distribution CAPMI

\[ E(R) - R^f = \alpha \text{cov}(R, R^w); \]
\[ E(R^w) - R^f = \alpha \sigma^2(R^w); \]

where:
\[ R^w = \text{gross return on total "wealth" portfolio; and} \]
\[ R^f = \text{gross risk-free return} \]

---

**Variables Not in the Value Functional; Other Limitations**

We must note here that the value functional in Equation 4-13 on page 60 does not include any utility-producing variables other than consumption from returns on investments within the current-time expectation. This and other aspects of the equations have important implications that bear noting:

1. Any variables that consumer-investors actually use to anticipate their future consumption (such as labor earnings, macroeconomic policies, investments in businesses not in the investment set) should be included in the information available to them.

2. There is an embedded assumption of “no shifts in the investment opportunity set,” and also identical and independent return distributions, and a non-variant, risk-free interest rate.\(^77\) There is no restriction on the kind of investment available.

3. There is no labor income, a clearly false assumption about the majority of consumer-investors.

4. The quadratic value function implies a highly unlikely view of potential losses.

Chapter 5

Traditional Theories of Value

by Patrick L. Anderson,

with contributions by Manav Garg

An Introduction

This chapter describes important applications of Mathematics to problems in Finance. The focus of this section is the mathematics rather than the application. However, in a few instances we briefly provide advice on use and avoidance of misuse of specific formulas.

Additionally, we reprint seminal works that define, or represent, certain standards of value. We also include guidance statements that describe the proper methodology to observe in practical valuation tasks, as stated by authorities in the field. Such authorities include authors, government agencies, and professional societies. Note that all such statements are prone to revision over time, and practitioners should periodically check to see if newer versions have been adopted.

Let us start by noting that very rarely does one encounter perpetual cash flows, constant growth rates, continuous-time payments, or known discount rates for investors. Yet, many of the formulas listed below presume that one or more of these conditions. Use all of these formulas with that caution in mind.
SECTION 5.1. A Note on Conventions; Discrete & Continuous Time

The conventions used to describe cash flows and present values by authors in Finance, Mathematics, and Economics often differ. It is worth noting these at the beginning of this Appendix.

- Most Finance texts use discrete time, in which payments or receipts can be neatly indexed by the time they are made. For example, \( C_i = C_0 \cdot (1 + r) \).

- A common convention for discrete-time Finance is that payments are received at the end of the period, and that the relevant time period is a calendar year unless stated otherwise. Note that, while it is certainly possible to include an immediate payment in a net present value calculation, the NPV of a payment today is identical to the amount of the payment, as \( (1 + r)^0 = 1 \).

- Many Economics and Mathematical Finance texts use continuous time models, in which terms involving Calculus expressions like \( \frac{dV}{V} \) or \( dX = \mu dt + \Sigma dz \) are used.

- Note that, when using integral calculus aggregate returns over continuous time, the integration normally starts at \( t=0 \). The integral of (continuously-compounded) discounted returns from zero to \( T \) will be approximately equal to the sum of the (discrete-time) discounted sum of returns during the time period from one to \( T \), if the effective interest rates used for the continuous-time and discrete-compounding are equivalent.

Examples of the difference in continuous-time and discrete-time present value formulas are illustrated in, for example, Section 5.3. “The Workhorse Net Present Value Algorithm.” In advance of that, we have the following useful formula to approximately convert annual, discrete-time interest rates to continuously-compounded rates:
**SECTION 5.2. Algorithms for Calculating Historical Growth Rates**

Consider a series of values indexed by time (this could also be represented as a vector of values), for which some process is causing the values to systematically grow (or shrink) over time. The process may include random elements, and the values are measured periodically and may be subject to measurement errors or missing data. We want to know the underlying secular trend, if any, in the data series. Algorithms to calculate the historical growth rates are presented below.

**a. The Arithmetic and Geometric Growth Rates**

There are two common arithmetic methods for calculating historical growth rates: arithmetic and geometric.

**Geometric (Compounded) Growth Rate**

The geometric average assumes a compounding over time. For data measured on an annual basis, the geometric mean of growth is called the Compounded Annual Growth Rate (CAGR). The following formula assumes that all elements of the vector of values are positive, and that the calculation is for an annual time series beginning with $V_t$ and running through the terminal value $V_T$.

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### Discrete-time Equivalent (Effective) Interest Rate

**EQ 5-1**

\[
    r_{\text{effective}} = \left( 1 + r \right)^{\Delta t} - 1;
\]

**where:**

- $r_{\text{effective}}$ = effective annual interest rate;
- $r$ = continuously-compounded annual interest rate;
- $e^r = \exp(r)$ = base $e$ of natural logarithm, raised to the $r$ power.

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March 29, 2013

*Solutions Manual and Reference for The Economics of Business Valuation*
A common application is to evaluate the growth in revenue for a firm from the beginning \((t=0)\) year through a terminal year:

\[
g = \left( \frac{V_T}{V_0} \right)^{\frac{1}{T-t}} - 1 \quad T > t, V \neq 0
\]

**Compounded Annual Growth Rate**  \( \text{EQ 5-2} \)

**CAGR in Revenue**  \( \text{EQ 5-3} \)

\[
g_{\text{rev}} = \left( \frac{\text{Rev}_T}{\text{Rev}_0} \right)^{\frac{1}{T-t}} - 1
\]

**Arithmetic Growth Rate**

The Annual Average Growth Rate (AAGR) is an arithmetic average of past per-period (annual) growth rates.

\[
g_i = \frac{V_i - V_{i-1}}{V_{i-1}}, \quad \overline{g} = \frac{1}{n} \sum_{i}^{T} g_i
\]

**b. Advice**

As historical review of revenue and profits in a firm is an essential part of estimating the market value of the firm, the topic of the metric used for that purpose is worth some attention. The following observations are based on practical experience in valuing firms as well as mathematical knowledge:
• The more volatile the series is, the farther apart these two growth rates will be.
• If the historical revenue or profits have been very volatile, the CAGR metric effectively ignores the volatility by calculating a compounded growth rate that would be the same whether there was a smooth path between the start and end points or not. The AAGR, on the other hand, takes into account this volatility by averaging over the entire path of growth.
• Although both these metrics appears to be “first moment” statistical measures (meaning describing the central tendency of a random distribution), the AAGR also captures some of the “second moment” (the variation in that distribution).
• The difference between the two metrics is often pronounced in smaller and start-up firms.
• Either metric will not make sense when the value for the initial period is close to zero.
• The CAGR is closer to the mathematical ideal of a constant growth rate. Many theoretical models in Economics assume constant growth rates, and log-linear functional forms make such assumptions convenient. However, data on actual firms are often limited to a small number of periods, where the AAGR may be a better indicator.
• Naively using the historical growth rate (measured either way) as a predictor of future growth is a classic mistake in business valuation. Beware of precisely measuring the past, while ignoring the future!78

Advice from Other Experts

The topic has been extensively discussed in relation to historical data on stock market returns, such as those reported in the Ibbotson Stocks Bonds Bills & Inflation references from various years (e.g., Ibbotson [2004, 2010]). No consensus exists.

Damadoran (2002) recommends the CAGR over the AAGR in business valuation. Abrams (2001) devotes an entire chapter to arithmetic and geometric means, and notes that the AAGR incorporates more available data. The Ibbotson firm has suggested that arithmetic means are better for business valuation purposes (Baran, [2002], Hitchner (2006, p. 167).

78 See also the discussion of revenue forecasting in Section 14.3. “Practical Task 1: Forecasting Revenue.”
although the volume of essays edited by William Goetzmann & Roger Ibbotson (2006) is more circumspect.

Section 5.3. The Workhorse Net Present Value Algorithm

The net present value (NPV) algorithm is a workhorse tool in Finance and Economics. Note that NPV is an algorithm, and may or may not produce an amount that represents the market value of any underlying asset.

a. Definition of NPV

The NPV algorithm requires these elements:

1. A sequence of real numbers, which may be negative or positive.79
   These numbers must be certain (fixed) at the time of the calculation of NPV. If the numbers are uncertain, then the NPV algorithm does not work; you may instead be able to calculate an expected net present value. See section 2.5 below.

2. A time index that orders the numbers in terms of sequential periods.

3. A base time period, usually the “zero” period.
   The net present value is defined for a particular period. Thus, the NPV at time \( t=0 \) of a certain stream of cash flows will often be different from the NPV at time \( t=1 \).

4. A discount rate.
   Normally, the discount rate is a positive rational number between zero and one. However, the algorithm will work for negative discount rates. The discount rate is usually fixed while the cash flows are var-

---

79 The set of real numbers (a mathematician might say “field”) includes integers, ratios (see below), and other numbers that have decimal representations (such as \( \pi \) and the square root of two). Another important quality is that real numbers include the limits of sequences that begin with real numbers. Most numbers in use in finance, accounting, economics, and business are real numbers, though not all. A real number is distinguished from an imaginary number, such as the (imaginary) square root of negative one, and from a complex number, which contains an imaginary component.

A subset of real numbers is rational numbers, which are numbers that are the ratios of integers. Note that all integers (including counting numbers such as 1, 2, 3) are also rational numbers, because an integer is a number that is divisible by one.
ied. However, this is due more to custom and data limitations than to fundamental economics.

\[ \text{NPV discrete intervals} \quad \text{EQ 5-5} \]
\[ \{C\} = [C_1, \ldots, C_T]; \text{ all } C_i \text{ are known, finite real numbers.} \]
\[ \text{NPV}(\{C\}) = \sum_{i=1}^{T} \frac{C_i}{(1 + d)^i} \]
\[ = \sum_{i=1}^{T} \beta^i C_i; \]

where:
\[ 0 < d < 1; \]
\[ \beta = \frac{1}{1 + d} = \text{gross discount rate}. \]

\[ \text{b. Continuous and Discrete Discounting} \]
The equations above use the discrete form favored in the finance and accounting literature. Below we show equivalent formulas of the type often seen in mathematical finance and economics, which involve continuous functions and integral calculus.

\[ \text{NPV Continuous Compounding} \quad \text{EQ 5-6} \]
\[ NPV(\{C\}) = \int_{t=0}^{T} e^{-\rho t} C_i \, dt \]

where:
\[ \{C\} = \text{series of payments over time}; \]
\[ C_i = \text{payment received at time } t; \]
\[ 0 < \rho < 1; \]
\[ \rho = \text{net discount rate (continuously-compounded)}. \]
Present Value Formulas: Continuous Compounding

Present value of single amount $K$ at time $t$:  
$$PV(K, r) = Ke^{-rt}$$

Present value of stream of earnings $\{K(t)\}$ over interval $t = 0...T$:  
$$PV\{\{K(t), 0, T\}\} = \int_0^T K(t)e^{-rt}dt$$

Present value of stream of earnings $\{K(t)\}$ at time $s$ over interval $t = s...T$:  
$$PV\{\{K(t), s, T\}\} = \int_s^T K(t)e^{-rt}dt$$

---

Section 5.4. Expected Net Present Value

a. The Expected NPV: Simple Discrete Case

Using the expectation from statistics, and the net present value from finance, we can identify the expected net present value algorithm. This algorithm is deceptively simple to state as the net present value of a series of expectations of the random cash flows $\{C_i\}$, discounted at the net discount rate $d$:

$$E_0[\text{NPV}\{\{C\}\}] = E_0\left\{\sum_{i=1}^L \frac{C_i}{(1 + d)^t}\right\} = \sum_{i=1}^L E_0(C_i)\left\{\frac{1}{(1 + d)^t}\right\},$$

where:

$\{C\} = [C_1, ..., C_L]$; all $C_i$ are random, bounded real numbers.

$E_0(\cdot)$ is the expectation operator, conditional on information available at time $t = 0$;

$\ d = \ $net discount rate, known and constant;

$0 < d < 1.$
This simplification results from the assumptions that the discount rate $d$ is fixed over time, so can be pulled out of the expectation; and the expectation is clearly defined.

b. Expected Value of a Function

The following is the more general case of the expected value of a function $g(x)$ of a random variable $X$:

\[
E[g(x)] = \int g(x) f(x) dx;
\]

\[x = \text{random variable}; \; g(x) \text{ continuous function of } x; \; f(x) = \text{probability density function for } x.\]

In general, the expected value of a function of $X$ is the function’s outcomes weighted by the probability density of $X$.

Section 5.5. Formulas for Annuities and Perpetuities

a. Future Values and Interest

For an account with a value at time $t$ of $S_t$, where the (constant, positive) per-period interest rate is denoted $r$, we have the following difference and compound interest equations:

\[
S_t = S_{t-1} + r(S_{t-1}) = (1 + r)S_{t-1} = R \cdot S_{t-1}. \]

\[S_t = S_0(1 + r)^t.\]

This is the same mathematics but a different notation than the formulas in Section 5.3. “The Workhorse Net Present Value Algorithm.” Using this notation, the following is the net present value equation:
Net Present Value of Future Value  \[ S_T = S_0 (1 + r)^T \]

Here, \( S_T \) is the future value, and \( S_0 \) is the present value.

---

b. Cash Flow Annuities

An annuity is a contract to pay a certain amount each period for a set of periods \( \{1, 2, \ldots, T\} \).

Present Value of an Discrete-Periods Annuity  \[ A_{1 \rightarrow T} = \frac{C}{1 + r} + \frac{C}{(1 + r)^2} + \ldots + \frac{C}{(1 + r)^T} \]

\[ = C \left[ \frac{1 - (1 + r)^{-T}}{r} \right] \]

---

c. Perpetuities

A perpetual series of constant cash flows is known as a perpetuity. Its present value (at a discount rate of \( r \)) is the following:

Present Value of a Perpetuity  \[ A_{1 \rightarrow \infty} = \frac{C}{1 + r} + \frac{C}{(1 + r)^2} + \ldots + \frac{C}{(1 + r)^T} + \ldots \]

\[ = \frac{C}{r} \]

where cash flow is constant.

The present value of a perpetuity, using a continuously-compounded interest rate, is shown in EQ 5-14.
Note that an equivalent result for continuously-compounded and annual-compounding can be achieved if the effective interest rates are equivalent; see EQ 5-1 and EQ 5-13.

\[
\text{PV (annuity) paying } C \text{ from } t = 1\ldots N: \\
A_{1+N} = C \frac{(1-e^{\rho N})}{r};
\]

\[
\text{PV (perpetuity) paying } C \text{ from } t = 1\ldots \infty: \\
A_{1+\infty} = \frac{C}{e^\rho - 1};
\]

where:
\(\rho\) = continuously-compounded interest rate;
\(C\) = payments received at the end of each period.

If \(e^\rho - 1 = r\):
\[
A_{1+\infty} = \frac{C}{r} = \frac{C}{e^\rho - 1}.
\]

d. **Constant Growth Model**

If we assume a continuously-compounded net interest rate, the future value of an initial sum \(K\) placed in an interest-bearing account, and the present value of such a future value, are given by the following equations:

\[
FV = Ke^{rt} = K^+;
\]

\[
PV = K^+ e^{-rt};
\]

\(0 < r < 1, \ K > 0, \) continuous compounding.
e. The “Gordon Growth” Formula; Variations; Limitations

If we have a base cash flow \( C_0 \) that grows perpetually at a constant rate \( g \), and a constant discount rate \( d \) with which to discount it, we have the very useful net present value formula known as the “Gordon Growth” formula. The derivation makes use of the fact that the ratio \( (1+g)/(1+d) \) can be expressed as \( a \), and that \( 1-a^{(n+1)} = (1-a)*(1+a+a^2+...+a^n) \).

\[
PV_{GG} = NPV \left( C_1 + C_2 + \cdots + C_i + \cdots \right) \\
= \frac{C_0 \cdot (1 + g)}{(1 + d)} + \frac{C_0 \cdot (1 + g)^2}{(1 + d)^2} + \cdots; \\
\text{where} \\
C_1 = C_0 \cdot (1 + g); \\
0 < g < 1, \ 0 < d < 1, \ g < d.
\]

Note that the formula assumes the first cash flow is received at the end of the first period, where it has already grown from the base \( t=0 \) amount. If you have a forecast of the \( t=1 \) amount (and the other factors listed below are fulfilled), you can use the alternative formula.

\[
PV_{GG} = C_0 \left( \frac{1 + g}{d - g} \right)
\]

\[
PV_{GG} = \frac{C_1}{(1 - d)}; \\
\text{where} \\
C_1 = C_0 \cdot (1 + g).
\]

The Gordon Growth formula can be used to model stock prices, using the (very strong) assumptions of constantly growing dividends and a known discount rate:
Advice

The Gordon Growth formula is another workhorse of applied finance. Unfortunately, it is a workhorse often drafted for inappropriate duty. Note the following very strong, and infrequently reasonable, assumptions:

1. Perpetual cash flows
   
   Note that this rules out changes in institutional factors such as dividend policy, tax rates, tax regime, or other factors, unless offset by a change in the earning power of the company.

2. Perpetual growth rate in cash flows
   
   It is improbable (some would say impossible) that a security or company could grow forever at a rate faster than GDP. Thus, any use of a growth rate that is faster than trend nominal GDP should be carefully considered; it is suspect.

3. Known, and constant, discount rate
   
   On this, the Gordon Growth formula shares the common difficulty of other methods.

For this reason, the Gordon Growth formula should not be used as a primary tool for valuation. Instead, it is useful as a rule-of-thumb, formula for comparison, or terminal value to a series of cash flows that have already been discounted for multiple periods.

For an interesting historical note on this formula (including the assertion that the “Gordon Growth” formula did not originate with Gordon [1962]), see Rubinstein (2006, pp. xii, 143).
SECTION 5.6. Income Subject to Termination Risk

a. Common and “Jump” Processes

The most common stochastic processes encountered in finance are those that generate outcomes have some central tendency. The “normal” distribution is an ideal example of this; other distributions (such as the log-normal) have a shape that “looks like” a squeezed, lopsided, or flattened bell curve.

However, some events are better characterized by a probability distribution that “jumps” from time to time, and otherwise stays constant. For example, consider the number of times that high-quality loans default, contracts with trusted counterparties are breached, franchises are terminated, or that lightning strikes. You know that, sooner or later, these events will occur. However, especially if the event is uncommon, it is probably not well modeled by a normal-looking distribution.80

b. Poisson Distribution

The Poisson distribution is often an appropriate model for such risks. This statistical distribution is close to that of a binomial distribution in which the number of trials is very high, and the probability of success in each trial is low. The binomial is often used to model events that have close to a 50% chance of occurring, such as a coin flip. The Poisson is typically used in studies of errors, breakdowns, queuing behavior, and other phenomena where the chance of any one subject facing a specific event is small, but where the number of subjects is large. The parameter lambda indicates the “mean arrival rate” of the unusual event; for example, a default risk of 2% in any given year could be modeled with a Poisson distribution with lambda=.02.

A Poisson process is governed by the following probability density function:

80 This has implications in many fields, including finance. In particular, if there are not a wide variety of securities that create a risk-reward frontier, or that could, under additional assumptions, be expected to have normally distributed returns or risk characteristics, many of the nice, standard conclusions of modern portfolio theory are undermined.
Poisson Distribution

\[ P(x) = \frac{e^{-\lambda} \lambda^x}{x!} \]

where:
\[ x = 0, 1, 2, 3, \ldots; \]
\[ 0 < \lambda < 1. \]

Note that the Poisson is a discrete probability distribution; it provides positive probabilities only for integers \( x = 0, 1, 2, \ldots \).\(^{81}\)

**c. Use in Valuation**

The use of the Poisson distribution is well-established in industries such as insurance, where jump risks are common. However, it has been largely ignored in business valuation. Anderson (2004) points out the usage of the distribution for termination risk inherent in business forms such as franchises, and in contract situations involving brands or suppliers.

**SECTION 5.7. Fair Market Value and Other Elements to Consider in Traditional Business Valuation**

Below we include reprint seminal works that define, or represent, certain standards of value from authorities including authors, government agencies, and professional societies. Note that all such statements are prone to revision over time, and practitioners should periodically check to see if newer versions have been adopted.

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\(^{81}\) The probability density function shown can produce non-integer numbers, but these are probabilities of certain integer values, not the values themselves. Random Poisson numbers always produce integers. For example, a monte carlo run of random Poisson numbers with \( \lambda = 0.25 \) produced 100 numbers: mostly zeroes, some ones, and one two. The sum of all 100 numbers was 21; \( 21/100 \) is close to the mean arrival rate of \( \lambda = 0.25 \).
Exhibit 5-1. Code of Federal Regulations: Fair Market Value

The fair market value is the price at which the property would change hands between a willing buyer and a willing seller, neither being under any compulsion to buy or to sell and both having reasonable knowledge of relevant facts. The fair market value of a particular item of property includible in the decedent's gross estate is not to be determined by a forced sale price. Nor is the fair market value of an item of property to be determined by the sale price of the item in a market other than that in which such item is most commonly sold to the public, taking into account the location of the item wherever appropriate.

Source: Code of Federal Regulations (for US Code Title 26), Sec. 20.2031-1 Definition of gross estate; valuation of property, subsection (b).

Exhibit 5-2. Professional Summary of Fair Market Value Elements (Mercer)

Section 20.2031-1(b) of the Estate Tax Regulations (section 81.10 of the Estate Tax Regulations 105) and section 25.2512-1 of the Gift Tax Regulations (section 86.19 of Gift Tax Regulations 108) define fair market value, in effect, as [1] the price at which the property would change hands [2] between a willing buyer [3] and a willing seller [4] when the former is not under any compulsion to buy and the latter is not under any compulsion to sell, [5] both parties having reasonable knowledge of the relevant facts. Court decisions frequently state in addition that [6] the hypothetical buyer and seller are assumed to be able, [7] as well as willing, to trade and [8] to be well informed about the property and [9] concerning the market for such property. [parenthetical numbers added]

Source: IRS, Mercer (1999); similarly stated in many other references.

Exhibit 5-3. Revenue Ruling 59-60 (1959)82

SEC. 4. FACTORS TO CONSIDER.
.01 It is advisable to emphasize that in the valuation of the stock of closely held corporations or the stock of corporations where market

82 Note: This ruling was later “modified” by RR 65-193, and “amplified” by RR 77-287, RR 80-213, and RR 83-120. However, these modifications and amplifications did not change the listed essential elements of a valuation estimate for US income tax purposes.
quotations are either lacking or too scarce to be recognized, all available financial data, as well as all relevant factors affecting the fair market value, should be considered. The following factors, although not all-inclusive are fundamental and require careful analysis in each case:

(a) The nature of the business and the history of the enterprise from its inception.
(b) The economic outlook in general and the condition and outlook of the specific industry in particular.
(c) The book value of the stock and the financial condition of the business.
(d) The earning capacity of the company.
(e) The dividend-paying capacity.
(f) Whether or not the enterprise has goodwill or other intangible value.
(g) Sales of the stock and the size of the block of stock to be valued.
(h) The market price of stocks of corporations engaged in the same or a similar line of business having their stocks actively traded in a free and open market, either on an exchange or over-the-counter.

Source: IRS; Original text provided by TaxLinks; found at: http://www.taxlinks.com.

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**Exhibit 5-4. Professional Summary of Valuation Elements (Anderson)**

1. Accounting Information
   a. Historical sales and revenue
   b. Cost and profitability
   c. Capital structure and discount rate
   d. History and form of business operation
   e. Any material, peculiar conditions
2. Economic Assumptions
   a. Growth assumptions for company
   b. Economic conditions in the industry
   c. Conditions in geographic trade area
   d. Value of franchises, licenses, and intangible assets
   e. Market transactions for similar businesses
3. Management Policies
   a. Management and management policies
   b. Important business plan assumptions

Factors to Be Considered When Doing a Business Valuation

Several factors need to be analyzed when conducting a business valuation. These include firm-specific factors as well as external variables such as the state of the economy and the condition of the industry. These external factors define the environment in which the firm operates. Firm-specific factors reflect the unique aspects of the firm as it operates in this economic environment.

Note: Gaughan then describes in separate sub-sections the following factors: firm-specific factors, economic factors, industry factors, and financial analysis. He also lists the elements of IRS RR 59-60 as a reference.

Source: Gaughan (2004), chapter 8.

On a larger scale, other factors shown in the following list must be considered before beginning the pathway shown in [a flow-chart figure in the original text showing the steps in a financial analysis].

1. **Macroeconomic Factors.** The economy,...
2. **Microeconomic Factors.** Industry analysis, industry valuation ratios, intra-industry competition, typical use of debt in the industry, customers, suppliers.
3. **Firm-Specific Factors.** Market share, historic financial characteristics, debt ratio, diversification, customers, suppliers, competitive position in the market.
4. **Selection of Comparable Companies...**
5. **Determination of Market Comparables for Target Company...**

All of these factors should at least be considered in preparing a valuation report, even if some of the factors to not play an important part in the analytical process.

a. Appraisal Practices

In the following exhibits we include guidance statements that describe the proper methodology to observe in practical valuation tasks, as stated by authorities in the field.

Exhibit 5-7. Professional Appraisal Standards for Federal Agencies

USPAP 2008–2009
Standards Rule 9-3

In developing an appraisal of an equity interest in a business enterprise with the ability to cause liquidation, an appraiser must investigate the possibility that the business enterprise may have a higher value by liquidation of all or part of the enterprise than by continued operation as is. If liquidation of all or part of the enterprise is the indicated premise of value, an appraisal of any real property or personal property to be liquidated may be appropriate.

Standards Rule 9-4

In developing an appraisal of an interest in a business enterprise or intangible asset, an appraiser must collect and analyze all information necessary for credible assignment results.

(a) An appraiser must develop value opinion(s) and conclusion(s) by use of one or more approaches that are necessary for credible assignment results.

(b) An appraiser must, when necessary for credible assignment results, analyze the effect on value, if any, of:

(i) the nature and history of the business enterprise or intangible asset;
(ii) financial and economic conditions affecting the business enterprise or intangible asset, its industry, and the general economy;
(iii) past results, current operations, and future prospects of the business enterprise;
(iv) past sales of capital stock or other ownership interests in the business enterprise or intangible asset being appraised;
(v) sales of capital stock or other ownership interests in similar business enterprises;
(vi) prices, terms, and conditions affecting past sales of similar ownership interests in the asset being appraised or a similar asset; and
(vii) economic benefit of tangible and intangible assets.

Comment on (i)-(vii): This Standards Rule directs the appraiser to study the prospective and retrospective aspects of the business enterprise and to study it in terms of the economic and industry environment within which it operates.
(c) An appraiser must, when necessary for credible assignment results, analyze the effect on value, if any, of buy-sell and option agreements, investment letter stock restrictions, restrictive corporate charter or partnership agreement clauses, and similar features or factors that may influence value.

(d) An appraiser must, when necessary for credible assignment results, analyze the effect on value, if any, of the extent to which the interest appraised contains elements of ownership control and is marketable and/or liquid.

Comment: An appraiser must analyze factors such as holding period, interim benefits, and the difficulty and cost of marketing the subject interest.

Equity interests in a business enterprise are not necessarily worth the pro rata share of the business enterprise interest value as a whole. Also, the value of the business enterprise is not necessarily a direct mathematical extension of the value of the fractional interests. The degree of control, marketability and/or liquidity or lack thereof depends on a broad variety of facts and circumstances that must be analyzed when applicable.


Usage note: Hitchner (2003, chapter 10) discusses this standard at length. Hitchner notes that compliance with USPAP is required for transactions under the authority of federal agencies including the Federal Reserve Board, FDIC, Office of Thrift Supervision, and Office of the Comptroller of the Currency. Hitchner also notes that the IRS has not adopted USPAP.

However, a subsequent transitional guidance statement (IRS Notice 2006-09) from the IRS states that an appraisal that meets the USPAP “substance and principles” will be “treated as having been conducted in accordance with generally accepted appraisal standards” standards under the Pension Protection Act of 2006.

Exhibit 5-8. Conduct Rule for Professional Appraisals for Federal Agencies

Conduct (Ethics Rule)

An appraiser must perform assignments ethically and competently, in accordance with USPAP.

An appraiser must not engage in criminal conduct.
An appraiser must perform assignments with impartiality, objectivity, and independence, and without accommodation of personal interests.

An appraiser must not advocate the cause or interest of any party or issue.

An appraiser must not accept an assignment that includes the reporting of predetermined opinions and conclusions.

An appraiser must not communicate assignment results in a misleading or fraudulent manner. An appraiser must not use or communicate a misleading or fraudulent report or knowingly permit an employee or other person to communicate a misleading or fraudulent report.

An appraiser must not use or rely on unsupported conclusions relating to characteristics such as race, color, religion, national origin, gender, marital status, familial status, age, receipt of public assistance income, handicap, or an unsupported conclusion that homogeneity of such characteristics is necessary to maximize value.

Source: USPAP (2008-09), Ethics Rule.


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**Exhibit 5-9. American Society of Appraisers BVS-IV Standards**

**AMERICAN SOCIETY OF APPRAISERS**

Business Valuation Standards

BVS-IV Income Approach to Business Valuation

I. Preamble

A. This Standard must be followed in all valuations of businesses, business ownership interests, and securities developed by all members of the American Society of Appraisers, be they Candidates, Accredited Members (AM), Accredited Senior Appraisers (ASA), or Fellows (FASA).

B. The purpose of this Standard is to define and describe the requirements for the use of the income approach in the valuation of businesses, business ownership interests, and securities, but not the reporting thereof.

C. This Standard applies to appraisals and may not necessarily apply to limited appraisals and calculations as defined in BVS-I, Section II.B.

D. This Standard incorporates the General Preamble to the Business Valuation Standards of the American Society of Appraisers.

II. The income approach

A. The income approach is a general way of determining a value indication of a business, business ownership interest, or security by using one or more methods through which anticipated benefits are converted into value.
B. Both capitalization of benefits methods and discounted future benefits methods are acceptable. In capitalization of benefits methods, a representative benefit level is divided or multiplied by an appropriate capitalization factor to convert the benefit to value. In discounted future benefits methods, benefits are estimated for each of several future periods. These benefits are converted to value by applying an appropriate discount rate and using present value procedures.

III. Anticipated benefits

A. Anticipated benefits, as used in the income approach, are expressed in monetary terms. Anticipated benefits may be reasonably represented by such items as dividends or various forms of earnings or cash flow.

B. Anticipated benefits should be estimated by considering such items as the nature, capital structure, and historical performance of the related business entity, the expected future outlook for the business entity and relevant industries, and relevant economic factors.

IV. Conversion of anticipated benefits

A. Anticipated benefits are converted to value by using procedures that consider the expected growth and timing of the benefits, the risk profile of the benefits stream, and the time value of money.

B. The conversion of anticipated benefits to value normally requires the determination of a capitalization factor or discount rate. In that determination, the appraiser should consider such factors as the level of interest rates, the rates of return expected by investors on alternative investments, and the specific risk characteristics of the anticipated benefits.

C. In discounted future benefits methods, expected growth is considered in estimating the future stream of benefits. In capitalization of benefits methods, expected growth is incorporated in the capitalization factor.

D. The capitalization factors or discount rates should be consistent with the types of anticipated benefits used. For example, pre-tax factors or rates should be used with pretax benefits, common equity factors or rates should be used with common equity benefits, and net cash flow factors or rates should be used with net cash flow benefits.

Source: ASA; found at http://www.bvappraisers.org/standards/bvstandards.pdf
b. Relevant Accounting Standards

Exhibit 5-10. Financial Accounting and Valuation: FASB
Statement of Financial Accounting Concepts No. 1 Highlights

Financial accounting is not designed to measure directly the value of a business enterprise, but the information it provides may be helpful to those who wish to estimate its value.

6. Financial statements are a central feature of financial reporting. They are a principal means of communicating accounting information to those outside an enterprise. Although financial statements may also contain information from sources other than accounting records, accounting systems are generally organized on the basis of the elements of financial statements (assets, liabilities, revenues, expenses, etc.) and provide the bulk of the information for financial statements. The financial statements now most frequently provided are (a) balance sheet or statement of financial position, (b) income or earnings statement, (c) statement of retained earnings, (d) statement of other changes in owners’ or stockholders’ equity, and (e) statement of changes in financial position (statement of sources and applications of funds).

41. Financial reporting should provide information about an enterprise’s economic resources, obligations, and owners’ equity. That information helps investors, creditors, and others identify the enterprise’s financial strengths and weaknesses and assess its liquidity and solvency.

Indirect measures of cash flow potential are widely considered necessary or desirable, both for particular resources and for enterprises as a whole. That information may help those who desire to estimate the value of a business enterprise, but financial accounting is not designed to measure directly the value of an enterprise.

Chapter 6

Applications

by Jeffrey Johnson

An Introduction

This chapter provides descriptions of the subject companies presented in Chapter 15 of *The Economics of Business Valuation* (EBV) along with in-depth examples showing applications of the value functional model. In addition to the subject company valuations, we perform valuations for a technology startup firm, and a rental property. These examples are intended to solidify the reader’s understanding of how the value functional approach can be applied in practice.

Section 6.1. Extended Descriptions of Valuation Subject Companies

In this section, we outline a profile of the three actual firms used in the valuation examples throughout EBV. We wanted to use actual firms, rather than fictitious entities with conveniently-assembled accounting statements, as test subjects for the multiple valuation techniques presented in this book.

As actual firms, they have histories, locations, management, competitors, reputations, and general management policies. The ownership is roughly representative of firms in the United States and Europe in the sense that the majority of them (two of the three) are privately-held, “small” firms, while the other is large and publicly traded.

Below we describe the basic operations, market position, key management figures, and other particulars of each firm from 2006 through 2010. We provide actual income statements and balance sheets for the two
privately-held firms. The names we assign to those firms are pseudonyms, and other identifying information has been obscured. Other than those minor alterations, the descriptions and information provided about each firm are accurate depictions.

a. S.H. Importers

*History of Enterprise, Industry, and Management*

In 1981, the founder began a sole proprietorship in Southern California, naming it S.H. Importers. The firm retails imported antique and reproduction antique furniture and furnishings primarily in Southern California. S.H. Importers sells primarily to interior designers (60%-70%) but also to individuals. The company also rents furniture and furnishings to the film and television industries and provides customized wood finishing and cabinetry to meet customer specifications. Furnishings include such items as: antique and Asian carpets, upholstery fabric, custom lamps, chandeliers, tableware, original artwork and statuary, lawn and garden furniture and ornaments, ethnic art, and decorating accessories.

The firm has an extensive inventory of “raw” and finished antiques acquired over the years, which assists S.H. Importers in meeting decorators’ numerous style requirements. A full cabinetry and re-finishing shop allows the company to customise furniture and finishes to meet decorators’ demands. This capability has also permitted the company to make its own reproduction furniture. The reproduction area has the potential to grow into a sizeable line of business on its own, but has not yet produced significant profits.

S.H. Importers has assembled a wide network of local and international suppliers over the years, which allows it to respond quickly to requests for specific pieces.

The company has approximately a half-dozen employees, although this number varies as is common in this industry. A number of the employees are part-time. There are two full-time employees, the founder and her son. The founder has been the primary manager and operator of the business for much of its existence. However, over the past decade the founder’s son has been trained as a manager and groomed as a successor owner. The son is now skilled at all aspects of the business and deals directly with customers and vendors. The company is, however, very dependent on the founder, for purchasing and merchandising. All of the
sales staff admit that the S.H. Importers owes a great deal of its success to the fact that customers are “buying Shirley’s taste”.

Ownership and Financial Performance

The founder of S.H. Importers still owns 510 shares, representing a 51% ownership interest. Her son owns the remaining 490 shares, representing a 49% interest. While the firm was very profitable in 2006, it has not earned sizable profits over the past three years. Historical income statements are shown below in Table 6-1.

**TABLE 6-1. S.H. Importers Income Statement, 2006-June 2010**

<table>
<thead>
<tr>
<th>Income</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
<th>June, 2010</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gross Sales</td>
<td>2,508,998</td>
<td>2,169,100</td>
<td>1,396,017</td>
<td>1,272,769</td>
<td>536,448</td>
</tr>
<tr>
<td>Discounts</td>
<td>(435,478)</td>
<td>(436,847)</td>
<td>(277,937)</td>
<td>(234,381)</td>
<td>(104,425)</td>
</tr>
<tr>
<td>Net Sales</td>
<td>2,073,520</td>
<td>1,732,253</td>
<td>1,118,080</td>
<td>1,038,388</td>
<td>432,023</td>
</tr>
<tr>
<td>COGS</td>
<td>789,639</td>
<td>652,947</td>
<td>415,680</td>
<td>396,809</td>
<td>158,314</td>
</tr>
<tr>
<td>Gross Profit</td>
<td>1,283,880</td>
<td>1,079,307</td>
<td>702,399</td>
<td>641,579</td>
<td>273,709</td>
</tr>
<tr>
<td>Movie Rental</td>
<td>38,173</td>
<td>50,714</td>
<td>16,690</td>
<td>11,450</td>
<td>12,748</td>
</tr>
<tr>
<td>Custom Carpentry</td>
<td>25,309</td>
<td>30,156</td>
<td>13,820</td>
<td>34,940</td>
<td>12,135</td>
</tr>
<tr>
<td>Other Income</td>
<td>5,266</td>
<td>5,569</td>
<td>2,924</td>
<td>2,512</td>
<td>5,365</td>
</tr>
<tr>
<td><strong>Total Income</strong></td>
<td>1,352,628</td>
<td>1,165,745</td>
<td>735,834</td>
<td>690,481</td>
<td>303,957</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Expense</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Depreciation</td>
<td>5,867</td>
<td>16,329</td>
<td>4,193</td>
<td>4,000</td>
<td>-</td>
</tr>
<tr>
<td>Rent</td>
<td>187,673</td>
<td>192,000</td>
<td>192,000</td>
<td>96,000</td>
<td>72,182</td>
</tr>
<tr>
<td>Property Taxes</td>
<td>12,712</td>
<td>12,964</td>
<td>13,206</td>
<td>13,672</td>
<td>6,858</td>
</tr>
<tr>
<td>Salaries &amp; Comm.Officers</td>
<td>49,042</td>
<td>45,500</td>
<td>50,500</td>
<td>34,250</td>
<td>9,000</td>
</tr>
<tr>
<td>Other</td>
<td>401,400</td>
<td>374,140</td>
<td>296,278</td>
<td>265,471</td>
<td>108,353</td>
</tr>
<tr>
<td>Total Salaries &amp; Comm.</td>
<td>450,442</td>
<td>419,640</td>
<td>346,778</td>
<td>299,721</td>
<td>117,353</td>
</tr>
<tr>
<td>SG&amp;A</td>
<td>284,843</td>
<td>292,118</td>
<td>230,770</td>
<td>193,971</td>
<td>132,071</td>
</tr>
<tr>
<td><strong>Total Expenses</strong></td>
<td><strong>941,536</strong></td>
<td><strong>933,051</strong></td>
<td><strong>786,948</strong></td>
<td><strong>607,363</strong></td>
<td><strong>328,464</strong></td>
</tr>
<tr>
<td>EBIT</td>
<td>411,092</td>
<td>232,694</td>
<td>(51,113)</td>
<td>83,118</td>
<td>(24,507)</td>
</tr>
<tr>
<td>Interest</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Shareholder</td>
<td>55,839</td>
<td>45,557</td>
<td>72,054</td>
<td>74,086</td>
<td>37,043</td>
</tr>
<tr>
<td>Other</td>
<td>1,776</td>
<td>1,303</td>
<td>808</td>
<td>285</td>
<td>-</td>
</tr>
<tr>
<td><strong>Total Interest</strong></td>
<td><strong>57,615</strong></td>
<td><strong>46,861</strong></td>
<td><strong>808</strong></td>
<td><strong>189</strong></td>
<td></td>
</tr>
<tr>
<td>Income Taxes</td>
<td>5,192</td>
<td>2,699</td>
<td>800</td>
<td>800</td>
<td>189</td>
</tr>
<tr>
<td><strong>Net Income</strong></td>
<td><strong>348,284</strong></td>
<td><strong>183,135</strong></td>
<td><strong>(124,775)</strong></td>
<td><strong>7,947</strong></td>
<td><strong>(61,739)</strong></td>
</tr>
</tbody>
</table>

The firm’s principals own the showroom and workshop facility, so S.H. Importers actually has no debt to speak of, except for Due-to-Principals, which is shown in Table 6-2 on page 91. This was especially beneficial
during the recent economic recession when several competitors have had to cease operations when unable to make rent payments or repay debt.

TABLE 6-2. S.H. Importers Balance Sheet, 2006-June 2010

<table>
<thead>
<tr>
<th>Assets</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
<th>June, 2010</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash</td>
<td>8,420</td>
<td>76,052</td>
<td>14,851</td>
<td>138,540</td>
<td>49,142</td>
</tr>
<tr>
<td>Accounts Receivable</td>
<td>242,862</td>
<td>153,365</td>
<td>18,003</td>
<td>43,077</td>
<td>44,109</td>
</tr>
<tr>
<td>Inventory</td>
<td>1,433,384</td>
<td>1,461,791</td>
<td>1,457,878</td>
<td>1,341,793</td>
<td>1,377,166</td>
</tr>
<tr>
<td>Other Current Assets</td>
<td>0</td>
<td>2,501</td>
<td>2,920</td>
<td>2,970</td>
<td>1,720</td>
</tr>
<tr>
<td>Total Current Assets</td>
<td>1,684,666</td>
<td>1,693,709</td>
<td>1,493,652</td>
<td>1,526,380</td>
<td>1,472,137</td>
</tr>
<tr>
<td>Fixed Assets</td>
<td>224,511</td>
<td>236,261</td>
<td>236,261</td>
<td>236,261</td>
<td>239,584</td>
</tr>
<tr>
<td>Net Fixed Assets</td>
<td>86,499</td>
<td>81,920</td>
<td>77,727</td>
<td>73,727</td>
<td>77,050</td>
</tr>
<tr>
<td>Total Assets</td>
<td>1,771,165</td>
<td>1,775,630</td>
<td>1,571,379</td>
<td>1,600,108</td>
<td>1,549,187</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Liabilities &amp; Equity</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Current Liabilities</td>
<td>88,938</td>
<td>64,167</td>
<td>56,379</td>
<td>22,650</td>
<td>95,368</td>
</tr>
<tr>
<td>Long-Term Debt</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Due to Principals</td>
<td>398,006</td>
<td>740,857</td>
<td>680,387</td>
<td>746,518</td>
<td>692,234</td>
</tr>
<tr>
<td>Other LTD</td>
<td>33,676</td>
<td>22,953</td>
<td>11,735</td>
<td>115</td>
<td>-</td>
</tr>
<tr>
<td>Total LTD</td>
<td>431,682</td>
<td>763,810</td>
<td>692,122</td>
<td>746,633</td>
<td>692,234</td>
</tr>
<tr>
<td>Total Liabilities</td>
<td>520,620</td>
<td>827,977</td>
<td>748,502</td>
<td>769,283</td>
<td>787,602</td>
</tr>
<tr>
<td>Shareholders Equity</td>
<td>1,250,545</td>
<td>947,653</td>
<td>822,878</td>
<td>830,825</td>
<td>761,585</td>
</tr>
<tr>
<td>Total Liabilities &amp; Equity</td>
<td>1,771,165</td>
<td>1,775,630</td>
<td>1,571,379</td>
<td>1,600,108</td>
<td>1,549,187</td>
</tr>
</tbody>
</table>

Note on Adjusted Post-Recession Income Statement

For the purpose of comparing valuation methods, we use a normalized post-recession income statement in one example valuation. This is based on the 2006 and 2007 actual income statements, with adjustments for distributed profits based on the assumption that professional managers could be substituted for the family members in the business, at a lower cost. These assumptions allow for a cleaner methodological comparison of a valuation of a profitable company with an option to expand. This example can be seen in full under “An Example Using a Markov Decision Problem (MDP) Method,” beginning on page 270 of The Economics of Business Valuation.

Market Transactions

There have been some intra-family transactions involving equity shares for this firm over the past several years. However, the management does
not believe these suggest the fair market value of the firm if it were to be sold without the owner.

The company does not have access to much market data on any truly comparable firms. However, it does have some information on transactions involving publicly-traded furniture stores or furniture assembly companies

b. A&A Consulting

History of Enterprise, Industry, and Management

A & A Consulting, offers business and public policy consulting services, as well as expert testimony. Founded in 1996, A & A Consulting has grown to nearly twenty full-time employees plus a number of part-time employees and contractors. It has two main offices: one in the capital city of a Midwestern state, and another in Chicago, Illinois. The company has also opened up satellite offices from time to time in other states, typically with a single part-time employee or contractor.

The firm has a diverse list of clients including governments, corporations, non profits, trade associations, and small businesses. Its experience includes markets throughout the Continental United States, as well as a small number of engagements in Alaska, Canada, Mexico, and the Caribbean. While the share of business outside the Midwest has been increasing over time, about a half of the company’s revenue comes from one state, with considerable variation year-to-year.

A & A Consulting operates in three practice areas, two of which have been managed by the company’s founder, who also serves as the Company’s CEO. The founder is a “key man” and brings in a significant amount of business. While this dependency poses a risk for A & A Consulting, he is working with other managers to cultivate talent to assist in this role over time. As part of this effort, new managers were appointed from within to oversee the two practice areas previously overseen by the founder and CEO. Currently, the firm has a well-developed position in a handful of niche markets, built on its excellent reputation.

A & A Consulting is one of the few consulting firms of its type in the U.S. that follows a quality assurance program. The company has a solid reputation for high-quality work. It tends to charge a somewhat higher price than its competitors in the local market, but a lower price than its
competitors in New York, Washington D.C., Boston, and Los Angeles. It is definitely a “boutique” consulting firm.

Ownership and Financial Performance

The primary shareholder in the company has been the founder for much of the past decade. Over the past several years, there have generally been 2 or 3 shareholders, with the founder owning about 90% of the equity. There is a written operating agreement which requires the company to establish certain written policies, and the company adopted a dividend policy that has resulted in regular distributions to members, even in the recent recession.

As shown below in Table 6-3, A & A Consulting has retained a healthy income statement despite the recession. Part of this is due to clients seeking out consulting services during tough times.


<table>
<thead>
<tr>
<th></th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Revenue</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Professional Billings</td>
<td>2,050,152</td>
<td>2,067,618</td>
<td>1,935,639</td>
<td>1,964,809</td>
</tr>
<tr>
<td>Reimbursed Expenses</td>
<td>82,868</td>
<td>58,408</td>
<td>41,463</td>
<td>55,247</td>
</tr>
<tr>
<td>Total Revenue</td>
<td>2,133,020</td>
<td>2,126,026</td>
<td>1,977,102</td>
<td>2,020,056</td>
</tr>
<tr>
<td>Discounts and Write-offs</td>
<td>(254,960)</td>
<td>(402,543)</td>
<td>(363,181)</td>
<td>(349,336)</td>
</tr>
<tr>
<td><strong>Total Revenues after Discount</strong></td>
<td>1,878,060</td>
<td>1,723,483</td>
<td>1,613,921</td>
<td>1,670,720</td>
</tr>
<tr>
<td><strong>Expense</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Payroll and Benefit Expenses</td>
<td>1,097,770</td>
<td>1,231,934</td>
<td>1,071,051</td>
<td>1,097,063</td>
</tr>
<tr>
<td>Insurance</td>
<td>2,857</td>
<td>15,297</td>
<td>8,396</td>
<td>11,393</td>
</tr>
<tr>
<td>Office Supplies &amp; Equipment Rental</td>
<td>55,106</td>
<td>59,589</td>
<td>50,826</td>
<td>55,815</td>
</tr>
<tr>
<td>Computer Software &amp; Data</td>
<td>43,402</td>
<td>40,563</td>
<td>38,996</td>
<td>34,555</td>
</tr>
<tr>
<td>Advertising and Marketing</td>
<td>118,282</td>
<td>191,355</td>
<td>150,633</td>
<td>197,556</td>
</tr>
<tr>
<td>Rent</td>
<td>67,073</td>
<td>106,211</td>
<td>119,434</td>
<td>133,266</td>
</tr>
<tr>
<td><strong>Total Operating Expense</strong></td>
<td>1,384,490</td>
<td>1,644,949</td>
<td>1,439,338</td>
<td>1,529,648</td>
</tr>
<tr>
<td>EBIT</td>
<td>493,570</td>
<td>78,534</td>
<td>174,583</td>
<td>141,072</td>
</tr>
<tr>
<td>Total Interest and Depreciation</td>
<td>(25,459)</td>
<td>(31,052)</td>
<td>(38,075)</td>
<td>(33,528)</td>
</tr>
<tr>
<td><strong>Taxes</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Personal Property</td>
<td>2,749</td>
<td>3,271</td>
<td>3,321</td>
<td>3,724</td>
</tr>
<tr>
<td>State Business Tax</td>
<td>12,177</td>
<td>9,322</td>
<td>15,014</td>
<td>8,314</td>
</tr>
<tr>
<td><strong>Net Income</strong></td>
<td>453,185</td>
<td>34,889</td>
<td>118,173</td>
<td>95,506</td>
</tr>
</tbody>
</table>
A & A Consulting has not acquired much long-term debt, but its balance sheet in Table 6-4 on page 95 does show a decline in their overall assets.

**Market Transactions**

There have been about a half-dozen market transactions involving equity shares for this firm over the past several years. These have implicitly valued the company in the range of $1.6 to $2.3 million, except for one single-unit transaction that suggested an implied company valuation of $3.6 million. The company’s management considers this an outlier and has accepted a series of valuation reports that, in recent years, suggest a market value of between $1.8 and $2.2 million.

The company does not have access to much market data on any truly comparable firms. However, it does observe some transactions involving other professional service firms that involve experts (such as management consulting), as well as direct analogues among a handful of publicly-traded expert witness firms.
c. ExxonMobil

Exxon Mobil is exclusively different from our other two example firms for several reasons. The first is its considerably larger size—Exxon Mobil is one of the world’s largest companies. That brings us to the second difference, which is that it is publicly traded. Third, it operates in the oil and gas industry, which is one of several industries that require some specialized knowledge, including complicated terminology, to make educated investment decisions.

<table>
<thead>
<tr>
<th>Assets</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Checking/Savings</td>
<td>487,295</td>
<td>228,322</td>
<td>277,909</td>
<td>207,343</td>
</tr>
<tr>
<td>Accounts Receivable</td>
<td>241,507</td>
<td>333,124</td>
<td>244,662</td>
<td>305,645</td>
</tr>
<tr>
<td>Allowance, doubtful accts</td>
<td>(50,000)</td>
<td>(60,000)</td>
<td>(75,089)</td>
<td>(57,500)</td>
</tr>
<tr>
<td>Prepaid State Business Tax</td>
<td>-</td>
<td>6,498</td>
<td>15,979</td>
<td>17,979</td>
</tr>
<tr>
<td>Total Current Assets</td>
<td>678,802</td>
<td>507,944</td>
<td>463,461</td>
<td>473,467</td>
</tr>
<tr>
<td>Fixed Assets</td>
<td>155,513</td>
<td>216,108</td>
<td>221,775</td>
<td>205,932</td>
</tr>
<tr>
<td>Accum. Deprec.</td>
<td>(68,738)</td>
<td>(109,717)</td>
<td>(139,450)</td>
<td>(145,516)</td>
</tr>
<tr>
<td>Net Fixed Assets</td>
<td>86,775</td>
<td>106,391</td>
<td>82,325</td>
<td>60,416</td>
</tr>
<tr>
<td>Other Assets</td>
<td>22,612</td>
<td>23,008</td>
<td>18,062</td>
<td>17,712</td>
</tr>
<tr>
<td>Accum. Amortization</td>
<td>(5,206)</td>
<td>(5,425)</td>
<td>(5,755)</td>
<td>(6,640)</td>
</tr>
<tr>
<td>Total Other Assets</td>
<td>17,406</td>
<td>17,583</td>
<td>12,307</td>
<td>11,072</td>
</tr>
<tr>
<td>Total Assets</td>
<td>782,983</td>
<td>631,918</td>
<td>558,093</td>
<td>544,955</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Liabilities &amp; Equity</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Current Liabilities</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Accounts Payable</td>
<td>3,005</td>
<td>5,472</td>
<td>-</td>
<td>780</td>
</tr>
<tr>
<td>Accrued Payable</td>
<td>35,665</td>
<td>19,585</td>
<td>25,582</td>
<td>63,242</td>
</tr>
<tr>
<td>Customer Retainers</td>
<td>230,438</td>
<td>183,722</td>
<td>200,839</td>
<td>177,065</td>
</tr>
<tr>
<td>Business Taxes Payable</td>
<td>16,120</td>
<td>16,523</td>
<td>12,477</td>
<td>13,966</td>
</tr>
<tr>
<td>W Note Payable</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>50,397</td>
</tr>
<tr>
<td>Accrual for Share Repurchase</td>
<td>-</td>
<td>-</td>
<td>79,788</td>
<td></td>
</tr>
<tr>
<td>Total Current Liabilities</td>
<td>285,228</td>
<td>225,302</td>
<td>318,686</td>
<td>305,450</td>
</tr>
<tr>
<td>Long Term Liabilities, W Note</td>
<td>-</td>
<td>-</td>
<td>89,640</td>
<td>62,996</td>
</tr>
<tr>
<td>Total Liabilities</td>
<td>285,228</td>
<td>225,302</td>
<td>408,326</td>
<td>368,446</td>
</tr>
<tr>
<td>Equity</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Beginning Capital</td>
<td>232,617</td>
<td>500,114</td>
<td>411,037</td>
<td>149,767</td>
</tr>
<tr>
<td>Members Distributions</td>
<td>(200,699)</td>
<td>(128,964)</td>
<td>(379,418)</td>
<td>(68,764)</td>
</tr>
<tr>
<td>Net Income</td>
<td>465,837</td>
<td>35,466</td>
<td>118,148</td>
<td>95,506</td>
</tr>
<tr>
<td>Total Equity</td>
<td>497,755</td>
<td>406,616</td>
<td>149,767</td>
<td>176,509</td>
</tr>
<tr>
<td>Total Equity and Liabilities</td>
<td>782,983</td>
<td>631,918</td>
<td>558,093</td>
<td>544,955</td>
</tr>
</tbody>
</table>
**History of Enterprise, Industry and Management**

Exxon Mobil is the world’s largest publicly traded international oil and gas company, as well as a direct descendent of John D. Rockefeller’s Standard Oil Company. Under the names Standard Oil, New Jersey Standard, Socony, Vacuum Oil, and Esso, Exxon Mobil has contributed significantly to key historical events. The Wright brothers used “Mobiloil” as one of the lubricants for their first flight at Kitty Hawk. The first of many Indianapolis 500 winners, Ralph De Palma, began using Mobil products in 1915. Five years later, Jersey Standard researchers produced the first commercial petrochemical—rubbing alcohol. Both Charles Lindbergh and Amelia Earhart used Mobiloil to make their solo flights across the Atlantic. In 1937, researchers produced artificial rubber, butyl, which is used today to create tires, surgical tapes, protective coatings and more.

Exxon Mobil continued to develop technology to discover oil fields and better refinery methods, which eventually became the industry standard for producing gasoline. Its synthetic automotive engine lubricant, which was introduced in 1974, is the world’s leading synthetic motor oil today. In short, over the past 125 years, Exxon Mobil has evolved from a regional marketer of kerosine within the U.S. to one of the best known brands in petroleum and petrochemicals worldwide.

As a leader in the energy and petrochemical industry, ExxonMobil operates facilities or markets products in most countries around the world and explores for natural gas and oil on six continents. By operating as an integrated company, ExxonMobil combines upstream activities with downstream operations, creating a presence in the two main sectors of the oil and gas industry.\(^\text{83}\) It is organized into a number of global operating divisions:

- Chemical division, based in Houston Texas;
- Downstream (marketing, refining, retail), based in Fairfax, Virginia; and
- Upstream (oil exploration, extraction), based in Houston Texas.

ExxonMobil is heavily invested in the overall energy industry, which requires long-term decisions that measure across decades and span multiple business cycles. As part of its social responsibility, ExxonMobil is actively involved in an industry-wide effort to update the “Oil and Gas

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\(^{83}\) The upstream sector involves exploration and production. Following the production phase, the downstream sector takes over through the point of sale.
Industry Guidance on Voluntary Sustainability Reporting”. This provides a series of performance indicators which help improve environmental, health and safety practices.

Additionally, ExxonMobil offers its customers economic incentives to go green. Prices are reduced when more efficient transportation is used, such as bulk deliveries and full truckloads. This reduces pollution, energy and trips, while saving the associated fuel costs.

Ownership and Financial Performance

Exxon Mobil is one of the largest publicly-traded companies in the world as well as its largest refiner. Its management system incorporates best practices developed all over the world and emphasizes early phases of project evaluation and execution. By incorporating rigorous reappraisal of projects, Exxon Mobil includes those learnings into future project designs and plans of execution.

Operations

Exxon Mobil tests its projects over a range of economic scenarios to “ensure that risks are properly identified, evaluated and managed”. By using this approach, management believes it allows for superior investment returns despite the obstacles created by business cycles. We show Exxon’s income statement in Table 6-5 on page 98.

Given the risks and challenges of the energy industry, Exxon emphasizes innovative technology and workforce safety as vital components to compete in the industry, as well as meet the growing demand for energy. The result of this management strategy and emphasis is high-performance products and improved manufacturing processes.

As shown above, Exxon Mobil has rebounded from its 2009 slump. As a whole, it is a solid investment, with 28 consecutive years of dividend-per-share increases. We show their balance sheet in Table 6-6 on page 99.
March 29, 2013

Page 99

TABLE 6-6. Exxon Mobil Balance Sheet (2006-2010)

<table>
<thead>
<tr>
<th>Assets</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
<th>2010</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current Assets</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cash &amp; Cash Equivalent</td>
<td>32,848</td>
<td>33,981</td>
<td>31,437</td>
<td>10,862</td>
<td>8,455</td>
</tr>
<tr>
<td>Accounts Receivable</td>
<td>28,942</td>
<td>36,969</td>
<td>25,272</td>
<td>27,645</td>
<td>32,284</td>
</tr>
<tr>
<td>Inventories</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Crude oil, products</td>
<td>8,979</td>
<td>8,863</td>
<td>9,331</td>
<td>8,718</td>
<td>9,852</td>
</tr>
<tr>
<td>Materials &amp; Supplies</td>
<td>1,735</td>
<td>2,226</td>
<td>2,315</td>
<td>2,835</td>
<td>3,124</td>
</tr>
<tr>
<td>Other Current Assets</td>
<td>3,273</td>
<td>3,924</td>
<td>3,911</td>
<td>5,175</td>
<td>5,269</td>
</tr>
<tr>
<td>Total Current Assets</td>
<td>75,777</td>
<td>85,963</td>
<td>72,266</td>
<td>55,235</td>
<td>58,984</td>
</tr>
<tr>
<td>Property, plant, equipment at cost, less accumul. depreciation &amp; depletion</td>
<td>113,687</td>
<td>120,869</td>
<td>121,346</td>
<td>139,116</td>
<td>199,548</td>
</tr>
<tr>
<td>Other assets, including intangibles, net</td>
<td>6,314</td>
<td>7,056</td>
<td>5,884</td>
<td>7,307</td>
<td>8,640</td>
</tr>
<tr>
<td>Total Assets</td>
<td>219,015</td>
<td>242,082</td>
<td>228,052</td>
<td>233,323</td>
<td>302,510</td>
</tr>
<tr>
<td>Liabilities &amp; Equity</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Current Liabilities</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Notes and loans payable</td>
<td>1,702</td>
<td>2,383</td>
<td>2,400</td>
<td>2,476</td>
<td>2,787</td>
</tr>
<tr>
<td>Accounts payable &amp; accrued liabilities</td>
<td>39,082</td>
<td>45,275</td>
<td>36,643</td>
<td>41,275</td>
<td>50,034</td>
</tr>
<tr>
<td>Income Taxes Payable</td>
<td>8,033</td>
<td>10,654</td>
<td>10,057</td>
<td>8,310</td>
<td>9,812</td>
</tr>
<tr>
<td>Total Current Liabilities</td>
<td>48,817</td>
<td>58,312</td>
<td>49,100</td>
<td>52,061</td>
<td>62,633</td>
</tr>
<tr>
<td>Long-term Debt</td>
<td>41,427</td>
<td>43,360</td>
<td>47,480</td>
<td>48,219</td>
<td>66,744</td>
</tr>
<tr>
<td>Other Long-term Obligations</td>
<td>11,123</td>
<td>14,366</td>
<td>13,949</td>
<td>17,651</td>
<td>20,454</td>
</tr>
<tr>
<td>Equity of Minority Interests</td>
<td>3,804</td>
<td>4,282</td>
<td>4,558</td>
<td>-</td>
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</tr>
<tr>
<td>Total Liabilities</td>
<td>105,171</td>
<td>120,320</td>
<td>115,087</td>
<td>117,931</td>
<td>149,831</td>
</tr>
<tr>
<td>Shareholders Equity</td>
<td>113,844</td>
<td>121,762</td>
<td>112,965</td>
<td>115,392</td>
<td>152,679</td>
</tr>
</tbody>
</table>

d. Market Transactions

ExxonMobil’s enterprise value, which is often used as an alternative to straightforward market capitalization was $403.7 billion as of June, 2011.\(^{85}\) Its market capitalization, or relative corporate size, on the same day was $402.41 billion.\(^{86}\) Despite being the world’s largest publicly traded company, ExxonMobil is considered a mid-cap, because it falls between $1 billion and $5 billion. Readers should note that while capitalization is a function of a firm’s stock price, it may not accurately reflect intrinsic value due to varying future expectations of investors.

Regardless, Exxon Mobil is making money for its investors: it distributed over $19.7 billion to its shareholders in 2010.

---

\(^{85}\) Enterprise value is often used as an alternative to straightforward market capitalization and is calculated as market cap plus debt, minority interest and preferred shares, minus total cash and cash equivalents.

\(^{86}\) Market capitalization is the number of common shares multiplied by the current price of those shares.
SECTION 6.2. Applications: Intermediate Results of Value Functional Examples in EBV

The example in this section is intended to provide the reader with the intermediate results for the valuation of A & A Consulting presented beginning on page 342 of EBV. This is done to guide the reader through the process of constructing and evaluating the model for the consulting firm in an effort to show practical applications of the value functional approach to business valuation.

a. A & A Consulting

A & A Consulting, an actual company included in Appendix C, “Description of Subject Companies,” in EBV faces an important investment decision, similar to that of S. H. Importers: the company’s founder is contemplating investing additional funds to expand the business. Here we use the value functional approach to assist the company’s management in making this decision. We base our analysis on actual income statements in recent years and assume the following:

1. Profits in the closely held consulting firm, which has been in existence for nearly twenty years in the same area of the Midwest, have been consistent for the past few years.

2. The company’s founder is contemplating investments in terms of hiring additional consultants and targeting new regional markets. The first investment may entail hiring new workers to expand production within the existing practice areas and locations. Further investment may lead to the creation of new practice areas, and a final investment may include opening additional locations in order to grow operations by expanding the company’s footprint.

3. Alternatively, the company could remain at its current scale.

4. The company cannot shrink below its current scale (state 1). Choosing not to invest additional funds in this state results in A & A’s operations remaining at current levels and shareholders receiving the current, lower level of distributed profits. Once the company has made an initial investment, however, future decisions by the founder not to invest carry the risk of the company’s operations slipping back to the levels of the previous state (see Figure 6-1).
**MDP Method: Event Tree and Action Space**

We structure the problem as follows:

- There are two possible actions for the firm: invest or do not invest. Investing has an immediate related cost, which the firm can avoid by choosing not to invest.
- There are four possible states of nature in which the firm can operate, each representing a different scale of operations and associated revenue. In this problem, the current (baseline) state is number 1.

To illustrate the potential movements among states, we use the event tree shown below in Figure 6-1. Note that the company is currently in state 1, with the potential to move up to a more desirable state or remain the same. Note also that the event tree does not allow the company to move more than one state up or down per time period.

**FIGURE 6-1. A & A Consulting’s Four States of Nature**


Use of the value functional approach requires a reward function or, in the case of a Markov Decision Problem (MDP) such as this, a reward matrix. This matrix explicitly identifies the reward associated with each
policy choice in each possible state of nature. With four states of nature and two available policy choices, this implies the creation of a 4x2 matrix.

Because we are solving an actual value functional problem using an actual company, we must carefully calculate these potential rewards. The method we use for this purpose includes the following steps:

1. We begin with an income statement for each state, reflecting the scale of operations and revenue in that state. We estimate these using the firm’s historical information, as well as projections for the future. For firms without comparable historical information for all states, one may be able to estimate the income statement for certain states using operating or financial ratios, plus management advice; using professional judgment to adjust from the current state; using comparable data on other firms; or using other methods.

2. In addition to the relevant lines from the income statement, the most important financial and operational revenues can be shown on the worksheet to ensure that the calculated rewards are reasonable given the assumed state and action.

3. Within the income statement for each state, we model the policy choice (in this case, discretionary investments) as it affects income and expenditures.

4. The benefits that accrue to the company’s shareholders (namely the founder) are typically the net proceeds to that person; these are the rewards of the reward matrix. In the case of a business valuation problem, care must be taken to ensure that the rewards here are only those for ownership of the firm, meaning that the same care should be taken in the value functional approach as in the income approach to ensure that any implicit owner compensation is taken into account.

We show the income statement for the company for each state, assuming no discretionary investment, in Table 6-7 on page 103.
TABLE 6-7. Income Statements for Each State: No Discretionary Investment

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<tbody>
<tr>
<td>Rev</td>
<td>$1,450,000</td>
<td>$1,700,000</td>
<td>$2,100,000</td>
<td>$2,500,000</td>
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<tr>
<td>Cogs (includes labor for consulting)</td>
<td>$1,087,500</td>
<td>$1,275,000</td>
<td>$1,575,000</td>
<td>$1,687,500</td>
</tr>
<tr>
<td>Subtotal: Gross Margin</td>
<td>362,500</td>
<td>425,000</td>
<td>525,000</td>
<td>812,500</td>
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<tr>
<td>Op exp</td>
<td>145,000</td>
<td>170,000</td>
<td>210,000</td>
<td>225,000</td>
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<tr>
<td>Net Capex and Other</td>
<td>72,500</td>
<td>85,000</td>
<td>105,000</td>
<td>125,000</td>
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<tr>
<td>Pretax Profit</td>
<td>145,000</td>
<td>170,000</td>
<td>210,000</td>
<td>462,500</td>
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<tr>
<td>Taxes</td>
<td>43,500</td>
<td>51,000</td>
<td>63,000</td>
<td>138,750</td>
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<tr>
<td>Net Profit (before discretionary investment)</td>
<td>101,500</td>
<td>119,000</td>
<td>147,000</td>
<td>323,750</td>
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<tr>
<td>Discretionary Investment (after tax effects)</td>
<td>-</td>
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<tr>
<td>Distributed Profits</td>
<td>$101,500</td>
<td>$119,000</td>
<td>$147,000</td>
<td>$323,750</td>
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</tbody>
</table>

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<tr>
<th>ratios:</th>
<th>pretax/revenue</th>
<th>net profit (before discretionary)/revenue</th>
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<tr>
<td></td>
<td>10%</td>
<td>7%</td>
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Reward Matrix

The resulting elements (one for each state-action pair) are assembled into the reward matrix shown in Table 6-8 on page 104. Observe that rewards shown in the first column are the results of the income statement calculations shown in Table 6-7 above.
TABLE 6-8. A & A Consulting Immediate Returns Given Each Action Possibility

<table>
<thead>
<tr>
<th>Reward Matrix</th>
<th>(2D: State x Action)</th>
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<tbody>
<tr>
<td></td>
<td>Actions</td>
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<tr>
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<td>Do Not Invest</td>
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<tr>
<td>state 1: Modest Size</td>
<td>101,500</td>
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<tr>
<td>state 2: First Expansion</td>
<td>119,000</td>
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<tr>
<td>state 3: Second Expansion</td>
<td>147,000</td>
</tr>
<tr>
<td>state 4: Bigger Scale and Profits</td>
<td>323,750</td>
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</tbody>
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Transition Matrix

Just as with the reward matrix, the value functional approach also requires a transition matrix. This matrix represents the probability of moving from a current state to a future state, taking into account both the firm’s policy choice and random events. There is a transition matrix associated with each action; it shows the likelihood of each possible next-period state given that action. Because the elements of a transition matrix are probabilities, they must sum to one for each current state. Following our convention of each current state being displayed in a row, this means each row of the transition matrix must sum to one. We show the transition matrix for this example valuation in Table 6-9 on page 105.

When examining the transition matrix, keep in mind that:

- We estimate the transition probabilities for each state and action pair. In this case, we believe the firm can only move up or down by one state per period.
- We recognize that the firm cannot expand operations without investing. This is shown among the transition probabilities through zero entries, which encapsulate the statement “the probability of expanding operations without appropriate investments is zero.”
- Conversely, the probabilities of increasing the scale of the operation are much higher when the action is “invest,” which is shown in the lower pane of the table.
• One way to quickly read the transition matrix is to observe how the probabilities tend to “lean” to one side, depending on the action.


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<tr>
<th>Current States</th>
<th>Do Not Invest</th>
<th>Invest</th>
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<td>Next-Period States</td>
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</tbody>
</table>
3. When the next period is reached, the investor-manager reexamines the state of the world and makes a new set of decisions based on the information available at that time.

We expect the investor-manager to choose a policy that maximizes the sum of the current reward and the expected discounted value of the firm in the next period. The premise of the value functional approach is that such an optimization provides the investor-manager with the value of the business to that particular investor in that particular state. Note that this problem, as defined, meets the conditions identified in Chapter 15 of EBV, and thus a solution does indeed exist.

**Solutions Using the Value Function Iteration Algorithm**

We display the results for the company’s investment opportunity below in Table 6-10. The table shows, for each state, the value of the firm in that state and the optimal policy.

**TABLE 6-10. Results for A & A Consulting’s Investment Opportunity**

<table>
<thead>
<tr>
<th>Summary of Outputs</th>
<th>Value</th>
<th>Best Policy</th>
</tr>
</thead>
<tbody>
<tr>
<td>State 1</td>
<td>$828,523</td>
<td>B</td>
</tr>
<tr>
<td>State 2</td>
<td>$893,576</td>
<td>B</td>
</tr>
<tr>
<td>State 3</td>
<td>$1,060,449</td>
<td>B</td>
</tr>
<tr>
<td>State 4</td>
<td>$1,388,519</td>
<td>A</td>
</tr>
</tbody>
</table>

We can immediately observe the following:

- As expected given the structure of the model, the estimated value of the firm is highest when the state is higher, meaning when revenues and operations are expanded.
- Recall that policy action B is to invest. In each state other than state 4 the recommended policy is to invest.

**Comparison with Naive Income Methods**

Although not illustrated here, for benchmark purposes we also calculated a value estimate using a naive income method. In particular, we calculated the value of a perpetuity of the immediate reward amount, using the assumed discount and growth rates. This is the equivalent of using the
common Gordon growth formula, or of simply extrapolating the results from the most recent period.

One can verify by hand calculation that, assuming a $101,500 profit in the current year and a discount rate of 15% the perpetual capitalization amount is approximately $775,000.

SECTION 6.3. Applications: Investment Decisions and Property Valuation

a. Startup Entrepreneur

In this example, based on a solutions template included with the Rapid Recursive® Toolbox license, we examine an investment decision from the unique perspective of the owner of a company that is not yet profitable. The owner is considering different investment options, including investing more of her own capital or accepting an investment from an outside source. Here we use the value functional approach to assist the entrepreneur in making her decision. We base our analysis on the advice of valuation experts at Anderson Economic Group, LLC, and on common market conditions around the year 2011 in typical USA markets. We also assume the following:

1. The firm is not yet profitable, and is developing an unproven, but promising technology application.
2. The entrepreneur is considering two potential investment sources: herself and a venture capitalist. An investment from either source will increase the company’s capital (increasing the likelihood of the firm earning a profit), but an investment from a venture capitalist would dilute the entrepreneur’s equity stake in the company (reducing her potential payouts), and also give preference in any future liquidation event to the venture capitalist over the entrepreneur.
3. In addition to accepting an investment from either of the sources mentioned above, the entrepreneur also has the option to do nothing or to exit the industry (shut down the company).

87 This example is based on an adaptation of the “Entrepreneur’s Valuation Problem” solutions template from the Rapid Recursive® Toolbox, © 2012 Supported Intelligence, LLC. Used with permission.
4. If the entrepreneur accepts an investment from the venture capitalist, the venture capitalist is given a 100% liquidation preference (in the event of a shut down as much of the venture capitalist’s investment as possible would be returned before the owner could recover any value).

5. The company currently has a prototype product, but has not yet submitted a patent application.

**MDP Method: State and Action Spaces**

We structure the problem as follows:

- There are four possible actions for the entrepreneur: do nothing, invest her own money, accept outside investment, or exit the venture entirely.

- There are seven possible states of nature in which the firm can exist, each representing a different stage of development and levels of the entrepreneur’s equity share. They are as follows:

  a. Bankrupt: The firm ceases operations and pays nothing to investors.
  b. Stopped: The entrepreneur has decided to shut down the firm. Assets are distributed to investors according to their equity stake in the firm and any established liquidation preferences.
  c. Prototype: The firm has developed a working prototype of their product.
  d. Prototype & Patent (PP) 80: The firm has developed a working prototype of their product and, if appropriate, has filed a patent for the product. In this state the entrepreneur has an 80% equity stake.
  e. Prototype & Patent (PP) 100: The firm has developed a working prototype of their product and, if appropriate, has filed a patent for the product. In this state, the entrepreneur has a 100% equity stake.
  f. Profit 80: The product has been launched in the market and is being sold profitably. In this state, the entrepreneur has an 80% equity stake.
  g. Profit 100: The product has been launched in the market and is being sold profitably. In this state, the entrepreneur has a 100% equity stake.

- The current (baseline) state is “Prototype.”
• Startup firms carry a high risk of failure (no return on investment). This is reflected both in the transition matrix in Table 6-13 on page 113, and the high discount rate\(^88\) (0.25) used in this model.

*Methods of Calculating Rewards*

Use of the value functional approach with a Markov Decision Problem requires a reward matrix. This matrix explicitly identifies the reward associated with each policy choice in each possible state of nature. With seven states of nature and four policy choices, this problem requires a 7x4 reward matrix.

The theoretical nature of this example allows us more flexibility (as opposed to the A & A Consulting example on page 100) in the creation of these potential rewards. Based on the advice of business valuation experts at Anderson Economic Group, LLC, we settled on the basic values outlined below in Table 6-11.

---

\(^{88}\) The discount rate is used to discount future values and implicitly captures the uncertainty of even receiving such values. Typical discount rates range between 0.05 and 0.15. As mentioned above, this model makes use of a discount rate outside of the typical range to account for the higher level of uncertainty associated with returns from startup firms.
TABLE 6-11. Startup Rewards Key Inputs

For each state and action pair, we compute the associated cost from the following equation:

\[ C_{ij} = v_{ci} \times VCFee + inv \times BaseInvAmount \]

where:
- \( i \) = state index
- \( j \) = policy index
- \( v_{ci} \) = 1 if outside investment has been accepted, 0 otherwise
- \( VCFee \) = Venture Capitalist Fee, defined above
- \( inv \) = 1 if entrepreneur invests her own money, 0 otherwise
- \( BaseInvAmount \) = Base investment amount, defined above

We calculate the value in each cell based on the following formula:
We calculate the entries of the 7x4 reward matrix using the formulas above. The results are shown in Table 6-12 on page 111.

\[ R_i(f) = e_i \times \pi_j - C_j(f) \]

where:
- \( i \) = state index
- \( j \) = policy index
- \( e_i \) = entrepreneur's equity stake in state \( j \)
- \( \pi_j \) = profit in state \( j \)

and
- \( C_j(i) \) is defined as above

**TABLE 6-12. Startup Entrepreneur Reward Matrix**

<table>
<thead>
<tr>
<th>Current State</th>
<th>Actions</th>
<th>Do Nothing</th>
<th>Re-invest</th>
<th>Accept20</th>
<th>Exit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bankrupt</td>
<td>$0</td>
<td>$0</td>
<td>$0</td>
<td>$0</td>
<td>$0</td>
</tr>
<tr>
<td>Stopped</td>
<td>$12,500</td>
<td>-$500,000</td>
<td>$12,500</td>
<td>$0</td>
<td>$0</td>
</tr>
<tr>
<td>Prototype</td>
<td>$0</td>
<td>-$500,000</td>
<td>$0</td>
<td>$0</td>
<td>$0</td>
</tr>
<tr>
<td>PP80</td>
<td>$775,000</td>
<td>$275,000</td>
<td>-Inf</td>
<td>$400,000</td>
<td></td>
</tr>
<tr>
<td>PP100</td>
<td>$1,000,000</td>
<td>$500,000</td>
<td>$1,000,000</td>
<td>$500,000</td>
<td></td>
</tr>
<tr>
<td>Profit80</td>
<td>$3,175,000</td>
<td>$2,675,000</td>
<td>-Inf</td>
<td>$4,800,000</td>
<td></td>
</tr>
<tr>
<td>Profit100</td>
<td>$4,000,000</td>
<td>$3,500,000</td>
<td>$3,200,000</td>
<td>$6,000,000</td>
<td></td>
</tr>
</tbody>
</table>

**Transition Matrix**

The value functional approach also requires a transition matrix, which represents the probability of moving from some current state (row) to some future state (column), given a certain action. Because the entries
of this matrix are probabilities, they must be non-negative and sum to one for each current state (row). When there are multiple potential actions, the transition matrix takes a third dimension to capture each of these. Thus the transition matrix has size $States \times States \times Actions$, or $7 \times 7 \times 4$ in this case. We show the transition matrix for this example in Table 6-13, “Startup Entrepreneur Transition Matrix,” on page 113 and 114.

While examining the transition matrix, keep in mind that:

- Each frame below represents the portion of the transition matrix associated with the identified action.

- We estimate the transition probabilities for each state and action pair. In this case, we believe the company can move to any of a number of future states from any given (current) state.

- We assume that the firm is unlikely to move to a higher state without some type of investment. This is shown in the first pane of the table by the fact that the transition matrix for the “Do Nothing” action is lower triangular (with just one exception, which represents a 20% probability that an outside investor exits when the firm is profitable).

- Conversely, the probability of moving to a more profitable state is much higher when the entrepreneur either makes or accepts an investment, which is shown in the middle two panes of the table.

- One can quickly read Table 6-13 by observing how the number of entries above the main diagonal changes depending on the action.
TABLE 6-13. Startup Entrepreneur Transition Matrix

<table>
<thead>
<tr>
<th>Current State</th>
<th>Next-Period State</th>
<th>Do Nothing</th>
<th>Re-Invest</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Bankrupt</td>
<td>Stopped</td>
<td>Prototype</td>
</tr>
<tr>
<td>Bankrupt</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Stopped</td>
<td>0.02</td>
<td>0.98</td>
<td></td>
</tr>
<tr>
<td>Prototype</td>
<td>0.02</td>
<td>0.98</td>
<td></td>
</tr>
<tr>
<td>PP80</td>
<td>0.02</td>
<td>0.1</td>
<td>0.88</td>
</tr>
<tr>
<td>PP100</td>
<td>0.02</td>
<td>0.1</td>
<td>0.88</td>
</tr>
<tr>
<td>Profit80</td>
<td>0.01</td>
<td>0.1</td>
<td>0.09</td>
</tr>
<tr>
<td>Profit100</td>
<td>0.01</td>
<td>0.39</td>
<td>0.6</td>
</tr>
<tr>
<td></td>
<td>Bankrupt</td>
<td>Stopped</td>
<td>Prototype</td>
</tr>
<tr>
<td>Bankrupt</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Stopped</td>
<td>0.01</td>
<td>0.19</td>
<td>0.8</td>
</tr>
<tr>
<td>Prototype</td>
<td>0.01</td>
<td>0.19</td>
<td>0.8</td>
</tr>
<tr>
<td>PP80</td>
<td>0.01</td>
<td>0.2</td>
<td>0.79</td>
</tr>
<tr>
<td>PP100</td>
<td>0.01</td>
<td>0.4</td>
<td>0.59</td>
</tr>
<tr>
<td>Profit80</td>
<td>0.2</td>
<td>0.4</td>
<td>0.8</td>
</tr>
<tr>
<td>Profit100</td>
<td>0.4</td>
<td>0.6</td>
<td></td>
</tr>
</tbody>
</table>
### TABLE 6-14. Startup Entrepreneur Transition Matrix (continued)

#### Structure of the MDP Value Functional; Existence of Solution

The structure of the value functional for this Markov Decision Problem and the proof for the existence of a solution exactly mirror those in the example above. See page 105 for a reminder of these arguments.

#### Solutions Using the Value Functional Iteration Algorithm

We display the results for the entrepreneur’s investment opportunity below in Table 6-15. The table shows, for each state, the value of the firm in that particular state and the optimal policy.
We can immediately observe the following:

- Recall that policy action A is for the entrepreneur to reinvest more of her own capital, policy action B is for the entrepreneur to accept outside investment, and policy action C is to do nothing. Before the firm has applied for a patent, the optimal action is to accept outside investment. Once a patent has been applied for or obtained, however, the best decision is for the entrepreneur to reinvest. This change comes from the lower probabilities of profit in the early pre-prototype stages, which make spreading the risk of loss to another investor more attractive than the entrepreneur simply sinking more of her money into the firm.

- As expected by the nature of the model, the estimated value of the firm to the entrepreneur is higher when the state is higher; meaning the product is better-developed and more profitable.

- The best policy is never to exit, or shut the company down, which implies that the value of the firm in operation is always higher than the exit payout.

**Comparison with Naive Income Methods**

Once can quickly see that, assuming the company is currently in the “prototype” state and earning no profits, a standard Discounted Cash Flow
(DCF) would imply a zero or negative value. This is quite different from the approximately $9 million predicted by the value functional model.

Although not illustrated here, for benchmark purposes we also calculated a value estimate using a naive income method. In particular, we calculated the value of a perpetuity of the immediate reward amount, using the assumed discount and growth rate. This is the equivalent of using a Gordon growth formula, or simply extrapolating results from the most recent period.

One can verify by hand calculation that, assuming a profit of $1,000,000 in the current year (profit in the PP100 state), an expected growth rate of 2%, and a discount rate of 25%, the perpetual capitalization amount is roughly $5,400,000. This value is significantly lower than the corresponding value from the value functional model because the naive income method does not account for the entrepreneur’s real options of investing and accepting outside investment.

b. **Landlord**

In this example, based on a solutions template included with the Rapid Recursive® Toolbox license, we examine a landlord’s decision to accept or reject an offer for tenancy. Each month, the landlord receives an offer for rent, which she must decide to accept or reject in hopes of a better offer next month. Here we use the value functional approach to assist the landlord in making her decision. We base our analysis on illustrative assumptions and assume the following:

1. For every month that goes by without an accepted offer, the landlord must pay a search cost (think of this as the cost of running an ad in the paper or online).
2. If an offer is accepted, the landlord receives the value of that offer, minus property taxes, each month until a separation occurs (lease ends or tenant leaves).
3. Once a separation occurs, the landlord immediately begins to face search costs and starts to receive offers again in the following month.
4. We assume this process continues indefinitely.

---

90 This example is based on an adaptation of the “Property Valuation Model” solutions template from the Rapid Recursive® Toolbox, © 2012 Supported Intelligence, LLC. Used with permission.
**MDP Method: State and Action Spaces**

We structure the problem as follows:

- There are two possible actions for the landlord: accept an offer or reject an offer.
- There are 22 possible states of nature in which the landlord may find herself. These include 21 states representing the values of each offer (from $2000 to $3000, inclusive, in $50 increments) and a final state for a separation.
- The landlord receives offers of each value with equal probability.
- The probability of a separation is 1/12, which represents an average of one separation per year after the landlord accepts an offer.
- Property taxes are $7,000 annually.

**Methods of Calculating Rewards**

Use of the value functional approach with a Markov Decision Problem necessitates the creation of a reward matrix. This matrix explicitly identifies the reward associated with each policy choice in each possible state of nature. With 22 states of nature and two policy options, this problem requires a 22x2 reward matrix.

The reward structure of this problem is rather straightforward. If the landlord rejects an offer, she must pay search costs and property tax. If an offer is accepted, the landlord earns the value of the offer less property taxes ($7,000 per year) each month until a separation occurs. Infinite costs are associated with the “accept” action during the separation state in order to force the landlord back into the market. Following this logic, we arrive at the reward matrix presented in Table 6-16 on page 118.

**Transition Matrix**

The value functional approach we use in this example also requires a transition matrix, which contains the probabilities of moving from some current state (row) to some future state (column), given a certain action. Because the entries of this matrix are probabilities, they must be non-negative and sum to one for each current state (row). When there are multiple potential actions, the transition matrix takes a third dimension to capture each of these. Thus the transition matrix has size $States \times States \times Actions$ or 22x22x2 in this case. We show the transition matrix for this example in Table 6-17 on page 119.
While examining the transition matrix, keep in mind that:

- Each frame below represents the portion of the transition matrix are associated with the identified action.
- We estimate the transition probabilities for each state and action pair. In this case, we believe the landlord moves to any of the possible offers with equal probability when an offer is rejected, and faces a 1 in 12 chance of separation each month after accepting an offer.
TABLE 6-17. Landlord’s Transition Matrix

**Structure of the MDP Value Functional; Existence of Solution**

Recall the general structure of a value functional equation for a landlord:

1. The landlord must decide how to manage her property, including whether to accept or reject offers for rent.
2. This decision affects the reward earned by the landlord and the value of the property at the beginning of the next period.
3. When the next period (offer) is reached, the landlord reexamines the state of the world and makes a new set of decisions based on the information available at the time.
The argument for the existence of a solution to this problem is a simple extension of that for the first example presented above. See page 105 for a reminder of this proof.

**Solutions Using the Value Functional Iteration Algorithm**

We display the results for the landlord’s rental opportunity in Table 6-18 below. The table shows, for each state, the value of the firm in that particular state and the optimal policy.

**TABLE 6-18. Results for a Landlord’s Rental Opportunity**

<table>
<thead>
<tr>
<th>Separation</th>
<th>Value</th>
<th>Best Policy</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>$522,504</td>
<td>A</td>
</tr>
<tr>
<td>2050</td>
<td>$522,504</td>
<td>A</td>
</tr>
<tr>
<td>2100</td>
<td>$522,504</td>
<td>A</td>
</tr>
<tr>
<td>2150</td>
<td>$522,504</td>
<td>A</td>
</tr>
<tr>
<td>2200</td>
<td>$522,504</td>
<td>A</td>
</tr>
<tr>
<td>2250</td>
<td>$522,504</td>
<td>A</td>
</tr>
<tr>
<td>2300</td>
<td>$523,020</td>
<td>B</td>
</tr>
<tr>
<td>2350</td>
<td>$523,599</td>
<td>B</td>
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We can immediately observe the following:

- As expected given the nature of this model, the estimated value of the rental property is higher when the state (offer) is higher.
• The best policy involves a reserve value. Recall that policy action A is to reject the offer and policy action B is to accept. The landlord should reject any offer at or below this value and accept any offer above. In this example, we can clearly see that the landlord’s reserve value is $2,250.

Comparison with Naive Income Methods

Although not illustrated here, for benchmark purposes we also calculated a value estimate using a naive income method. In particular, we calculated the value of a perpetuity of the yearly reward for an offer of $2,500 using the assumed discount and growth rates. This is the equivalent of using a Gordon growth formula, or simply extrapolating results from a representative period.

One can verify by hand calculation that, assuming an offer of $2,500 per month ($23,000 yearly when taxes are accounted for), a growth rate of 2% and a discount rate of 6% the perpetual capitalization amount is $609,500. This value is significantly higher than the corresponding value from the value functional approach because the naive method does not account for the probability of separation, search costs incurred after a separation, and the likelihood of receiving lower offers.
Chapter 7

End of Chapter Questions and Hints

by Patrick L. Anderson

An Introduction

For each chapter in the *The Economics of Business Valuation* (EBV), the author prepared a set of questions, which are shown below. The author also developed hints to several questions, to further develop the material and provide a starting point for some readers.

SECTION 7.1. Review Questions for Chapter 1: Modern Value Quandaries

Consider the following questions:

1. Contemplate further the following time periods and economies:
   • During ancient times (say, anytime before the fall of the Western Roman Empire around 400 AD), is it reasonable to say that “businesses” were the primary employers and wealth-producers, if you consider farms to be businesses?
   • During the Middle Ages in Europe, guilds grew in importance and power for many artisans, craftsmen, and merchants. Do you consider the individual members of such guilds, if they practiced their profession, to be businesses?
   • In the current age, consider the Western European countries with very large public sectors, along with Canada, the United States, and

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India with smaller, but still large, public sectors. Are private sector businesses not the primary employers in any of them? Are they the primary “wealth-creators”? Does your answer depend on how you define “wealth” in relation to work done by the government?

**HINT**

*Think about the meaning of “wealth” here. “Wealth” implies more than just income; it implies physical capital (house, car, clothing), savings or investments, capability and willingness to spend money on ordinary and essential things as well as some non-ordinary items. Human capital (extended schooling, training, time with parents and mentors) could also be considered part of “wealth.”*

Ask yourself if government and nonprofit entities can create this kind of wealth, without funding it by taxing private sector workers and businesses.

2. Think about the intellectual history distilled in the first section of the chapter (especially pages 3-5), and the assertion (in Quandary 1) that “mainstream economics ignores the firm.” Is it possible that business is as important as claimed earlier in the chapter, but the institution of a business is “ignored” by mainstream economics? Does your answer change if you consider the quotations from Walras and Dreze?

3. If you have taken a recent course in Economics or Finance, pull open the textbook for that course.
   - Does the textbook discuss primarily the incentives facing the consumer, or the incentives facing the business manager or entrepreneur?
   - If it is a textbook with “finance” in its title, are the subject businesses typically publicly-traded corporations, banks, governments, and pension funds; or small privately-held companies and entrepreneurs?
   - If you have taken both “Economics” and “Finance” courses, do you agree with the assertion in EBV Quandary 7 that the literatures in these two fields have diverged? Are there good examples (besides this book, of course) that attempt to cross over between the two?

• If you know the owner of a small business, ask him or her if the primary motivation for managing the firm is “maximizing profits.” Ask further if the business ever makes an investment or hires a worker (or the reverse) even if it results in a short-term reduction in profits.

• If you have a book from the question listed above, check quickly to see if the assertion “entrepreneurs do not maximize profits” is supported, contradicted, or ignored in that text.

HINT

If you are taking a formal class in Economics or Finance, consider how many of the examples in the textbook, or discussed in class, assume the truth of the assertion that “businesses maximize profits.”

Also consider how many bona fide business owners have spoken in the class, or are quoted in the textbooks and assigned writings.

5. Do you think the claims in “Elements of a New Model of the Firm and Its Value” on EBV page 12, are really necessary?

HINT

If you have a typical “Introduction to Economics” textbook, you probably see an explanation of the neoclassical economics briefly described in The Economics of Business Valuation. Try comparing this with a classical text, such as Wealth of Nations, and then with a contemporary-era book by, say, George Gilder or Thomas Sowell. Note how these authors discuss the moral foundations of the economy; is this same grounding apparent in the textbook?

For extra credit (or extra punishment, depending on your point of view), now read some of the (occasionally polemical) articles from people that write about economics and current events, such as Paul Krugman in the New York Times or Bill O’Reilly, author of The No Spin Zone and a television commentator in the United States. Then look at the editorial page of The Wall Street Journal, the columnists in Forbes, or the columnists in The Economist; and then a few columns in The Atlantic, Rolling Stone, and The New Yorker.

Ask yourself, are all these people talking about the same topic?

For added perspective (if you have access), read some of the editorial
commentary in the business-oriented newspapers in India, Singapore, Turkey, or Brazil. How do they talk about the economy, the moral foundation of society, and the problems society faces? Do you see similarities between the arguments about the economy and society in the United States with those in these rapidly-developing countries?

SECTION 7.2. Review Questions for Chapter 2: Theories of Value

Consider the following questions:

1. Consider the ten principles of value listed in this chapter. Can you think of others? Are any redundant?

*HINT*

Don’t be shy here!

The popular business press, as well as the academic press, is full of valuation theories, formulas, strategies, wise sayings, foolish examples, and a certain amount of inane commentary. Think hard about distilling this morass down into a small set of distinct principles of valuation.

2. “Principles” of value versus “methods” of valuation

Other authors list numerous methods of valuing companies, even within the same general approach. For example, Pablo Fernandez\(^\text{92}\) lists 8 methods and 7 theories just to “value companies by cash flow discounting”! In EBV, Chapter 13 and Chapter 14, the author cites variations of the traditional income approach that provide a gamut of acronyms alone (NCF-E, FCFF, 2-stage FCFF, etc.). Furthermore, this

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does not include the numerous methods and data sources that could be
used to create the discount factor used in all of these income methods.
Are these based on different theories or principles of value, or are they
different applications of the same theory or principle?

**HINT**

When is a formula a “theory” or “principle”? I would answer “rarely.”

*An equation is an equation; if it is a definition, then it is just that.*
*(For example: average = \( \text{sum}(X)/n \) just defines the word “average” in terms
of the other two variables and the operation “sum.”)*

*Many other “equations” are communications between persons engaged in
commerce, but are not theories or principles.* *(For example: A listing of items
on a restaurant bill, with a sum at the bottom, is on its face just a list and
a sum. However, the restaurant believes it is also a demand for payment!)*

*If an equation truly captures a specific natural or human behavior, and all
the elements of the equation are well-defined, then it might be a distillation
of a theory.* *(For example, \( E=mc^2 \) is a distillation of a theory in physics,
requiring of course explanations of every element in the equation.)*

3. **Principles of value and their usefulness**
   What principle would be the most important for you, if you were valu-
ing each of the following:
   • a car;
   • a warehouse full of inventory owned by your firm;
   • stock in a publicly-traded firm;
   • a partnership interest in a dentist office;
   • the rights to name (on earth) a never-visited star; and
   • the rights of first refusal to buy a parcel of land adjacent to your
   home or office?

4. **Distinctions among valuation theories**
   In particular, what are the distinctions between:
• the income and asset approaches, given that both rely upon accounting statements?
• the income and value functional approaches, given that both consider income to be valuable?
• the market approach and all others, given that estimating market value is the goal of most valuation efforts?
(You may want to look ahead at the chapters that discuss these particular methods before answering.)

5. What is a market?
Consider what we mean by the “market” in which fair market value is determined. Now answer the following questions:
• Is a large auditorium or public square filled with desirable goods, all listed with a selling price, a market if there are no buyers present?
• If the sellers were present, but removed their goods because they did not want to sell, would there still be a market?
• If few buyers and sellers were actually present, and all the goods were listed on an online auction service, would that constitute a market?

Section 7.3. Review Questions for Chapter 3: Failure of the Neoclassical Rule
Consider the following questions:

1. The $E(NPV)$ algorithm
   The author is harshly critical of the naïve use of $E(NPV)$ in valuation. After reading both its impressive intellectual pedigree, and the summary of its limitations, are you similarly critical, modestly skeptical, or entirely unconvinced?
   Look at a few finance, accounting, or economics books. Are the authors of these books skeptical of the “rule” as well? Do they accept it uncritically?

2. Failure, or just flaws?
Do you think that the $E(NPV)$ rule is fundamentally correct, and that the anomalies pointed out by the author merely reveal minor flaws? If so, describe the difference between “flaws” and “failure” of a theory that underlies both investment behavior and management decisions.

3. $E(NPV)$ and the financial press

Now look at the financial pages of an investment-oriented publication. Based on this evidence, do investors differ on discount rates, subjective probability distributions of future states, or investment time horizon? What evidence suggests this?

What does this tell you about the usefulness of the $E(NPV)$ tool? What does it tell you about the importance of the inputs to the $E(NPV)$ algorithm, as opposed to the algorithm itself?

**HINT**

One way to look at this problem is the following:

*If markets were complete and there was no arbitrage, then all prices would be enforced in the manner described in The Economics of Business Valuation in the chapter on no-arbitrage pricing in perfect markets. In such a (fantastical) world, there would be little need for investment advice, business periodicals, or wise men and women talking on television about “the markets.”*

Yet, there are.

*Now, pose the same thought experiment about a Modigliani-Miller world, in which the lack of arbitrage enforces a uniformity of prices (and returns) among securities in a “risk class.” In that world, do we need all these chattering pundits and effusive investment advice?*

4. $E(NPV)$ and “adjustments”

Suppose a very knowledgeable economist (or accountant, or financial analyst) tells you that all financial valuation comes down to the expected value of future “cash flow,” and that many valuation models fail because they fail to properly define cash flow, or fail to make the proper “adjustments” to the expected cash flow or the discount rate. Suppose the same expert shows you how models that failed miserably
to predict the market value of some business would have been accu-
rate, if only the cash flows and discount rates were properly adjusted.
Does this mean that $E(NPV)$ works, doesn’t work, or isn’t the funda-
mental basis of the expert’s valuation model?

5. Expert knowledge
Suppose the experience described in the previous question convinces
you that expertise, gained from experience in actual markets, greatly
assists a person in estimating the market value of a business.
What does this tell you about the $E(NPV)$ algorithm, the use of
“adjustments,” and the role of expertise in valuation?

6. Evidence from your decisions
Now, consider the major investment or management decisions you
have made recently. What factors other than $E(NPV)$ did you consider
in these decisions, if any? Did you consider, for example, the option to
wait? Asymmetrical risks? Strategic considerations involving other
persons, institutions, elections, etc.?
If you did consider these other factors, what does that tell you about
the “rule”?

SECTION 7.4. Review Questions for Chapter 4: The
Nature of the Firm
Consider the following questions:

1. Why do firms exist?
Select the most compelling theory and describe how it explains the
existence of firms. Should the number of firms be growing, or declin-
ing, based on recent economic, tax, and cultural trends?

2. Theories of the firm
Do you agree that the theories (such as the transaction cost theory, the
resource-based theory, and the value-added theory) fail to provide a
valid definition of the firm? If not, can you show how the preceding
theories distinguish between a *bona fide* firm and the example businesses posed in this chapter?

3. Tax treatment of corporations

Why would firms choose a “C corp” structure if it means a higher tax burden on income distributed to shareholders? Do you think the share of economic activity performed by C corps should be declining because of this? Are there offsetting advantages to the C corp form?

**HINT**

*After you answer this question logically, pose the same question in the matter a polemical author might do. (See the hints for Chapter 1 for a few suggestions.)*

4. The business value axiom: scale

Consider again the “bar of gold example.” What if, instead of four bars of gold and four people to guard them, the examples involved four tons of gold and forty people to guard them?

5. The business value axiom: financial assets

Re-consider again the “bar of gold example.” What if, instead of four bars of gold and four people to guard them, the examples involved four portfolios of financial option contracts, involving several million dollars in notional principal; and four accountants hired by the owner of the contracts to administer (but not “manage”) them. Would your answer change if the accountants were instead MBAs, economists, or soothsayers? If they “managed” the portfolio by using their professional judgement to buy and sell?

Would your answer change if the aforesaid employees also attempted to gain portfolio management business from other investors?

**HINT**

*The business value axiom is important to the theory of the firm and its value. Without this axiom, it is difficult to see what “value” a business might have that is separate from its physical assets and existing operations. Indeed, in a Modigliani-Miller world, it is these assets and operations that form the basis for stockholder’s investments in those firms. In a broader world, the*
human capital, ideas, motivations, management skills, people, intellectual property, and brand equity all matter as much, and often more.

Ask yourself a clarifying question: is pile of gold (or lead, or stock certificates) the same thing as a mutual fund, except for the size of a pile and the person watching it?

6. Definition of a firm

Do you agree with the definition offered here of a “firm”? If not, what examples can you cite of bona fide businesses that violate one of the premises? How would you redefine “firm”?

7. Institutional Requirements for businesses

Do you agree with the author’s assertion that institutions such as private property are necessary to organize and run firms? If not, why not? Can you cite a successful industry, or a sample of firms (not just products) that trade internationally and which do not rely on private property rights?

HINT

To help you answer this question—or challenge you after you did—read a few of the publications and authors listed above in the hints for EBV Chapter 1 “Theories of Value.”

8. Replicable business processes

Do you agree with this tenant as one required for a “firm”? If not, can you cite a firm without the capability to perform at least one business process repeatedly?

HINT

This is an important topic for people that want to manage business operations, or invest in them. It is even more important for people that want to start or lead them. Entire industries have fallen because they forgot the importance of this, while others rose. (The automobile industry in the last 50 years is a telling example of the importance of replicable business practices, and the costs of failing to maintain successful ones.)
9. Further examples of businesses

Consider the difference between two businesses: a fund that owns stocks and bonds for one investor, and an investment management company that serves a number of clients.

Consider the difference between two businesses: first, a consultant that is extremely smart, and performs a lot of services. This very smart person owns a small business entity that may employ a handful of people. Second, a consulting firm that has the same (small) number of people, none of whom are extremely smart, but who organize their work so that the employees of the firm can perform it even if the founder is not present.

Assume that all these entities are taxable business entities that perform good services for their customers, make an accounting profit, pay taxes, and pay their employees.

Which are “firms” under the definition adopted in this book? Which are “firms” from the perspective of a customer, the taxing authority, and employees? Is it possible that one group could see a business differently than another?

10. The business value axiom in practice

Return to the previous question. Which of these businesses would have value, if the current owner-operator was no longer present?

Section 7.5. Review Questions for Chapter 5: The Organization and Scale of Private Businesses

Consider the following questions:

1. Types of organization

What are the various types of business organizations under the laws of the United States of America and its States? Consider in your answer whether you would primarily distinguish between companies on the basis of sizes, for-profit or non-profit, tax filing status, or form of organization.
2. Dividends and organization
   A company issues stock and pays dividends to its shareholders. Do you have enough information to know what form it takes? If you know there are over 1000 shareholders, can you give a more precise answer?

   **HINT**

   Think about the laws in the United States regarding corporations, partnerships, and LLCs. If that doesn’t help, re-read the chapter.

3. “Small” businesses
   Do “small” businesses occupy a small, medium, or large role in the U.S. economy? Would a firm with $50 million in revenue and 250 employees be considered by you to be “small”? How about the national government in your country?

   **HINT**

   Be specific about what you mean by the size of the role. Is technology, innovation, cultural change, customer satisfaction…or just revenue important? It may also be useful to review this Companion Volume’s discussion—see Section 2.2., “Comparing U.S. Firms by Size.”

4. The firm and the monolithic entrepreneur
   What is the difference between a small shop headed by a single entrepreneur; a middle-sized company with several shops; and a national franchise of shops?
   What do you think of the assertion (generally attributed to Long and Bryant) that a closely-held firm is not a smaller version of a large corporation?

   **HINT**

   Do you have a difference of viewpoint from the author, Ronald Coase, or Long & Bryant? How about from a corporate finance textbook of your choosing?
5. Rationale of the firm

Why are businesses organized? Consider the theories presented in this chapter. Which is the most compelling?

6. Outsourcing and the firm

If individuals could purchase office supplies, rent office space, hire temporary employees, and contract with accounting and human resource agencies for the functions usually handled by the personnel and finance departments in a large firm, contract for manufacturing products overseas, and hire independent contractors to sell the product; are there other reasons to form a traditional business organization? What are they?

7. The firm and the team

Consider the social nature of man, and assume that there is an intrinsic desire of workers to join into social groupings of some type. What does this mean for the relative value of the following: (a) traditional companies that provide training, apprenticeship, and information to workers that generally live in the same area as the workplace; (b) traditional companies that provide all these benefits, but assign workers to remote duty that provides little social interaction for long periods; (c) companies that “outsource” as many services as possible to non-employees, and focus on providing competitive pay and benefits for the fraction of workers involved in the production of their services that are actually employees of that company?

Think about the wages demanded by workers (and contract workers) in companies in each of the categories above, and their productivity. Do you think there is likely to be a higher, or lower, average wage for workers employed by companies in category (c) above than in (a) or (b)?

**HINT**

Consider the well-publicized high pay and lavish benefits (including, apparently, multiple-cuisine free cafeterias for all workers) paid by some prestigious Silicon Valley firms, at least during years in which they were making a very large amount of money. Now consider that some firms have been publicly cutting back on lavish perks. What does this make you think about the answer to your question?
8. Value of firms in the United States

Based on the available information in this chapter, and any updated information available from sources such as the Federal Reserve Board or the Internal Revenue Service, do you think the estimates of the aggregate market value of private and public firms contained in this chapter to be reasonably accurate?

What would be a range of approximation for a “reasonably accurate” estimate for a number that is well into the $trillions and is not directly observable?

If you could improve on the estimate in this chapter, what data sources and methods would you use?

9. The C and S corps; taxes and value

Consider a firm that has been operating as a C corp, with publicly traded stock, and whose income is taxed at the company level. Assume that the firm typically distributes the majority of these earnings (after tax) to its stockholders.

Does the value of the firm’s equity change if it changes its form of organization to a partnership and withdraws from public markets? Can you identify factors that both benefit, and reduce the benefits, of ownership given such a change?

SECTION 7.6. Review Questions for Chapter 6: Accounting for the Firm

Consider the following questions:

1. Accounting and the Firm

Assume you are the CEO of a company with a one-time opportunity to earn large profits by acting quickly to take advantage of a new technology. Unfortunately, acting quickly would rule out the time-consuming tasks of creating timely accounting reports, and would not allow the investors to review your actions for at least six months. Your lawyer tells you that no law requires you to file these statements, only your promises to investors. Would you do it?
HINT

I can provide no hint for individual readers’ consciences. However, think about your past statements to your friends, co-workers, and family members when you read about a corporate leader going to jail (or getting bounced off a board, or getting slapped on the wrist) due to a lack of managerial controls. Does your answer match your behavior?

2. Abandoning the historical cost principle
   Continuing this example, assume that market conditions for raw materials, and the retail market for sporting goods, fluctuates during a one-year period. If you abandoned the historical cost principle for the supplier, manufacturer, and retailer, how would these same transactions be recorded?

3. FCFF and FCFE
   Why might cash flow to the firm as a whole be a better indicator of financial health than cash flow to equity? When would the reverse be true?
   Why would you rely more on these measures than on net income, and when would you abandon them in favor of other measures?

4. Accounting Manipulation
   Review again the section on ways to increase cash flow. If an accountant (or investment analyst) wanted to make a firm appear to be more valuable, can you identify ways that this could be done by manipulating the accounting statement? Once this was done in one period, what would investors learn in subsequent periods?
   Could the same manipulations be used to make a firm appear less valuable? What motivation might exist for a manager or investor to make a firm appear less valuable or profitable?

5. The “most important accounting principle”
   During the several years during which The Economics of Business Valuation was being written, the Great Recession occurred, as did disruptions in the world financial system that were sometimes called a “meltdown.” At that time, and afterwards, it became apparent that some accounting firms (as well as bond rating firms) had participated
in audits and ratings that left investors with an inflated view of the financial solvency of certain financial institutions during that time. Does this mean that accounting is more, or less important? Do you agree with the author (and the accounting authority he cites) that integrity is the most important principle of accounting?

**SECTION 7.7. Review Questions for Chapter 7: Value in Classical Economics**

Consider the following questions:

1. **Is the Labor theory really dead?**

   We argue in this chapter that the Labor Theory of Value is wholly inadequate as a theory of business valuation.

   Can you point to any examples of the following classes of people using an explicit or implicit labor theory of value in their pricing of good and services?
   - Dentists, doctors, or other Medical professionals
   - Authors
   - Tradesmen and women (including electricians, carpenters, computer repair technicians)
   - Lawyers and accountants
   - Economists
   - Lobbyists
   - Politicians

   If your answer to any of these is “yes,” does that mean the Labor Theory lives on? If so, with what scope and relevance?

2. **The undead Labor theory, continued**

   Read through a sample of investment commentaries on publicly-traded companies that are considering starting a new line of business, or acquiring another business to enter a new market or expand their presence in an existing one. Can you see any elements of the Labor theory of value? If so, do you think it is warranted in terms of a more accurate investment analysis?
3. “Sweat Equity” and the Entrepreneur

Many successful entrepreneurs develop value in their businesses partially by working very, very hard. This is sometimes called “sweat equity” in the United States.

Does the existence of sweat equity mean the labor theory of value is alive and well, at least with entrepreneurs?

HINT

Do you ever make broad pronouncements such as “those that work hard get ahead.” If so, are you a believer in the labor theory of value?

4. The Internet Stock Bubble

Many high-tech businesses, which appeared to rely upon new technology and the explosion of the Internet, were founded, capitalized by investors, and failed in or around the year 2000. Does the labor theory of value offer any insight into why many of these companies, which were sometimes derided as promising products that consumers never understood, failed the market test?

5. Adam Smith and free-market economics

Although Adam Smith’s conception of the labor theory of value is inadequate to describe the value of a business today, Smith’s writings on the role of the free market in promoting liberty and prosperity are as relevant today as they were in 1776.

Consider again the argument for property rights in Chapter 4’s “Institutional Requirements for a Firm” on page 55. Does the thought of Adam Smith (and other contemporaries, such as John Locke) have other meaning in today’s economy?

6. Adam Smith and the mixed economy

Adam Smith’s thoughts are still relevant. However, they are also still controversial. Consider whether your answer to any of the questions posed above would be the same in a “mixed economy” that consists of state-sponsored enterprises that operate alongside private firms in some markets and supplant them in others; where certain industries are dominated by state-regulated companies; and where wholly-pri-
vate companies often require permits, licenses, and allowances to operate.

**HINT**

For “mixed economy,” consider the case of Sweden, Germany, and the UK in the years before Reagan came to power in the 1980s. Now consider the US economy in the education and health care sectors; the US and European banking sectors, and Germany, France, and Italy. Which of these are “mixed” economies?

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**SECTION 7.8. Review Questions for Chapter 8: Value in Neoclassical Economics**

Consider the following questions:

1. **Definition of Value**
   
   What is value, to a neoclassical economist, to a classical economist, to a laborer paid by the hour, and to a philosopher?

2. **Gross rate of return and rate of accumulation**
   
   Recall Debreu's rate of accumulation. If a bond purchased today for $100 pays a coupon of $10 and has a principal value of $100, both of which are returned to the investor one year hence, what is the rate of accumulation? What is the gross rate of return? If someone asked you what the interest rate on the bond was, what would you say? Show how the various concepts are algebraically related.

3. **How “neo” is Neoclassical Economics?**
   
   Consider the triumphant intellectual innovations of neoclassical economics, including the use of “marginal” analysis, the price-quantity graph, and the notions of consumer welfare and utility. Now, just how useful is all that, if you have to run a business? Does any customer of that business actually talk about his or her “welfare” or “risk preference” or “utility”?
How relevant to new industries and technologies is neoclassical economics?

**HINT**

If you think neoclassical economics is flawed, with what would you replace it? Are there areas where you agree with the neoclassical synthesis?

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**Section 7.9. Review Questions for Chapter 9: Modern Recursive Equilibrium**

Consider the following questions:

1. Standard investment models and young people

   Standard advice given to young investors is to start saving for retirement early, yet many do not. Consider the following explanation: For most young people, the opportunities to gain skills, purchase usable assets like vehicles and houses, and other demands consume all of their current earnings. It often makes sense for them to purchase these assets (or build their “human capital” by staying in school) at that time in their life, and to start saving later when they are able to earn more money than they then require for living expenses.

   Do you agree with this as an explanation for low savings among young investors? Do it make failure to invest “rational” for many of these individuals?

   What, if anything, does this say about pricing of investment assets offered to young investors?

   **HINT**

   If you are trained as an Economist, think about this from an updated neoclassical perspective, where the consumer was assumed to have a utility function based on an actuarially-correct, von Neuman-Morgenstern (expected value maximizing), discounted for life-expected consumption set of preferences. Now:

   - Do you actually know anyone outside of your economics classes that knows what that means? Can you describe it in words to your friends? If not, what does that say about assuming that the broad mass of consum-
ers follow that kind of behavioral rule? Does this rule describe the behavior of anyone that you know?

• Alternatively, can you describe for the same people the notion of valuing something you can consume today more than the same thing you can consume tomorrow (or next week or month), meaning just choosing between consuming today and saving for tomorrow? Do you know anybody that behaves as if they follow that kind of pattern of behavior?

Have the prior examples given you an illustration of the difference between a recursive model of decision-making, and an expected value model?

2. The consumer’s savings problem

In what sense is the consumer’s savings problem really the consumer’s investment policy decision?

3. The “basic” pricing equation

Is the word “basic” in BPE a trifle heroic? Can you describe investment or asset purchase decisions that are not done using some form of the BPE?

4. Stochastic discount factor

What is so “stochastic” about the SDF used in, say, the purchase of “risk free” government bonds? A new car purchased using a loan? A home mortgage? Consider your answer from the perspective of both the buyer and seller (borrower and lender).

**HINT**

Recall that the SDF in this model is typically good for one period of time, which implies it may change over time.

This aspect is often not presented explicitly in mathematical finance texts, but it is important to observe. Any one-period model based on consumer preferences or market prices or state transition probabilities must, by nature of these assumptions, be a model that involves parameters that change over time.

5. Utility functions
In the preceding question, we get a clear answer only if we make a specific assumption about consumers. In terms of the “utility function” of microeconomics, what assumptions do we make about the following: consumers preference for more, rather than less; risk, rather than stability; cake, rather than pie?

6. The Bellman equation and $E(NPV)$

Consider a small business with good accounting records, in a stable location, with enough profit to remain a going concern. Can you describe the conceptual differences between valuing the store using these two approaches?

Assume that you know how to calculate the answer using both methods, and the parameters for the estimation were available to you, which would you choose, and why?

**HINT**

You may want to come back to this question after trying to use the recursive method in practice.

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**SECTION 7.10. Review Questions for Chapter 10: Arbitrage-Free Pricing**

Consider the following questions:

1. Weak Law of No Arbitrage
   
   Do you agree with the argument that produced the Weak Law of No Arbitrage?
   
   If not, how many mathematical arbitrage trades can you identify that occurred in real life? If so, what use is the “weak law”?

2. Risk-Neutral and Real-World Probabilities

   Have you ever bought a lottery ticket? Try to recall the subjective probabilities you assigned to winning. If you knew that most government-run lotteries in the US actually pay out only about 50% of the
ticket revenue in prizes, would you have bought more, less, or the same number of tickets?

3. Risk-Neutral Probabilities
   Have you ever observed a risk-neutral probability? How?
   What is the risk-neutral probability of rolling a “6” on a fair die, if nobody bets on it?

4. State Prices
   If there was an observable state price for the following events (a price of a security that paid $1 if, and only if, the event occurred one month hence), what do you think it would be?
   • “Good weather” for the July 4th weekend
   • A “white Christmas”?
   • A full moon on Halloween
   • The Tigers, Yankees, Red Sox, or Cubs win the pennant in baseball
   For each of these, who would buy such a security, and who would sell? How would the popularity of the individual sports teams affect the price? Why isn’t there a “full moon Halloween” security?

   **HINT**

   Think expansively about the existence, or lack thereof, of some type of contract, commodity, equity, or other traded asset that encapsulates some of the underlying risks involved. For example, “good weather on the 4th of July weekend” does affect demand for resort hotels, lake access, and beaches in the United States. However, how could you buy and sell a contract involving that demand?

5. State prices, risk-neutral probabilities
   Consider again the baseball teams, and assume a market of fans that bet on the outcome of the season. Would the market-clearing price of a bet on one team be based on the objective probabilities of the team winning, the team’s popularity, or both?
HINT

Gambling is irregularly prohibited, allowed, regulated, and ignored across much of the world. Therefore, there is evidence on this question. For a hint, think of the origin of the term “bookie” in the U.S., and how such a person arranged his or her financial transactions.

6. Incomplete Markets

The previous questions postulated a market for weather on holidays, and for pennant winners in baseball. If there are no state prices for the items in the previous examples, are these markets complete?

7. Completing the market

Suppose only the four teams mentioned above were in the league, and pennant securities did exist for two of them. Could you synthesize a security to insure against the risk that neither of them won the pennant? What if there were three such securities?

8. The benefits of arbitrage

Arbitrageurs are often derided as “speculators” and, since they make money by purchasing assets that are priced low and then re-selling them at a higher price, criticized for not producing any tangible benefit to society.

Consider the basketball example described on p. 148. Explain how Frank, Barb, and the community were benefited, or were hurt, by the actions of the arbitrageur in that market.

HINT

I offer no hint for the answer to the specific question. However, once you have answered it, consider the treatment of “middlemen” and “speculators” in the business press, in the popular press, and in the statements of politicians of various parties. Is there an accepted and universal view of arbitrageurs in society?

9. Partially complete markets

Consider equity interests in all the following businesses:
• Small dentist offices
• Franchised restaurants
• Car dealerships
• Companies with “penny stocks” and stocks traded infrequently, but listed on some exchange
• A large company that has filed bankruptcy, but continues to have stocks traded on a major exchange
• A closely-held partnership, such as a large accounting firm, in which there are over 1000 equity owners.
• A publicly-traded company of large size that is part of many large-cap mutual funds.
Which of these are traded in complete, incomplete, or partially-complete markets? For which markets, if any, could the “no arbitrage” assumption be justified?

10. Using the mechanism
For which of the categories above can you use the mechanisms of complete markets pricing, and how much of the mechanism can you use?

**HINT**

In addition to the obvious lack of a secondary market, consider carefully transaction costs, counter-party risk, and due diligence requirements. (This anticipates the next question.)

11. Transaction Costs
Choosing the category you found the closest to a complete market, what is the effect of transaction costs (such as commissions) on the price? On the “no arbitrage” assumption?

**HINT**

See hint above.
SECTION 7.11. Review Questions for Chapter 11: Portfolio Pricing Methods

Consider the following questions:

1. CAPM: Worth a Nobel Prize?
   This chapter contained specific critiques of the use or misuse of CAPM, largely based partially on the significant difference between the assumptions used to derive the model and the real-world conditions in which it is often applied.
   Can a theory of investment that technically “works” only within narrow confines provide a useful intuition outside those confines?
   The Nobel Prize committee—and decades of researchers—think the CAPM is quite important; how important do you think it is? How useful?

2. Factor Models; Macroeconomic Factors
   A wide range of macroeconomic factors have been suggested as relevant to investment asset returns, especially equity (stock) returns. Some authors have proposed “factor models” using the assumption that some such factors affect business income and therefore should also predict returns.
   Do you think any such factors should be added to an empirical asset-pricing model equation, if: (a) the purpose is to simply explain prices; or (b) the purpose is to provide investment advice; or (c) the purpose is to guide public policy?
   Does the existence of consistent explanatory power for some such factors mean the CAPM is wrong? Does your answer to this question make you second-guess your answer to the previous question?

3. State Variables and Factor Models
   In Chapter 15 “The Value Functional: Theory” we suggest state variables that appear to be relevant to an entrepreneur.
   Which of these, if any, would you include in a factor model? Would you include them in such models if they did not appear to improve the goodness-of-fit of a regression model?
Which of these would you recognize as valid state variables for a privately-held firm, but for which you would not expect to find empirical data?

What additional state variables would you suggest are relevant, and should any of these be considered for inclusion in a factor model explaining publicly-traded firm equity prices?

**HINT**

Think of this first theoretically, as if computational difficulty and access to data were not a concern. Then, think of the same question practically. You may want to look at a few of the examples for actual businesses, either in *The Economics of Business Valuation*, Chapter 17, or this Solutions Manual, Chapter 6.

Also consider the following question: in a standard DCF model, with much of the scenario driven from a forecast of trend growth in revenue, how many state variables are being considered? How many are explicit in the model? Can you disentangle the explicit state variable(s) from the subjective adjustment(s)?

### Section 7.12. Review Questions for Chapter 12: Real Options and Expanded Net Present Value

Consider the following questions:

1. **Real Options and Market Value: Publicly-Traded Firms**
   
   If a company is publicly traded, and is known to engage in heavy R&D, are its “real options” for technology and intellectual property incorporated into its stock price already?

2. **Replicating Portfolio**
   
   How would you attempt to construct a replicating portfolio for the following firms:
   
   - A start-up service firm with a very talented entrepreneur.
   - A start-up in another industry.
• A natural resources company that primarily sold raw materials.
• A natural resource company that primarily purchased raw materials.
• A well-established service firm.
• A firm that sells widely-purchased products in the U.S. and Europe that are produced in other countries, and that faces a small-probability chance of having its assets confiscated by a hostile government?

**HINT**

*Most of the methods traditionally proposed for dealing with “real options” involve this step. A critical test of those methods is the ability to form the replicating portfolio, or at least a mimicking portfolio. For at least the first example (the entrepreneur), I know of no way to replicate the risks and opportunities inherent in a motivated, talented entrepreneur.*

3. Real Options Techniques

What real options (or other valuation) techniques would you suggest for the various firms suggested above?

4. Real or financial options

Where would you draw the dividing line between real and financial options? Consider the “Brief History of Option Contracts” on page 170. How would you classify the common agricultural based forward-like contracts in early-modern Japan? Forward contracts today? Adjustable-rate mortgages? Personal guarantees on business loans?

5. Black-Scholes-Merton Assumptions

• Check to see that the underlying assumptions about stocks and (risk-free) bonds in the B-S-M formula (Equation 12.2) for pure financial options match those of the option valuation model of the firm (Equation 12.3).

• How applicable is the Black-Scholes formula to a real option? There are a number of widely-available spreadsheet templates and software programs that claim to estimate the value of real options, using a modification of the B-S-M or binomial model of financial option prices. Do you think these can estimate the value of a real
option in a market without a replicating portfolio?
What would you put into such a formula for “volatility” if the main
risk faced by the possible investment was an unlikely, but possible,
catastrophic failure?

6. Expanded Net Present Value (XNPV)
Do you think actual managers use the \( NPV \) rule rigidly when they see
option-like features in investment opportunities? Do you think they
should use the NPV rule?
What do you think about the use of an \( XNPV \) rule? How would you
write such a rule for a typical firm?
The concept of \( XNPV \) is one of adding real option values to the oper-
ating earnings-based value. Do you think that you should ever sub-
tract real options from the income-method estimate of operating
earnings-based value?
How would you combine (add, subtract, or ignore) the earnings from
current operations for a firm with the value from potential earnings
from real options, if the company was pursuing these real options and
absorbing the cost of them in their income statement?

\textit{HINT}

See the technical paper that describes valuation using different methods for
the oil & gas firm YPF, listed under Hints for Chapter 15.

7. Recursive decision making
Does the value functional method of addressing the “shutdown”
option for a plant actually require a recursive decision making
method? Couldn’t an analyst just calculate the likely state path in
advance, and management plan for the shutdown from that day for-
ward? (You may wish to consult other chapters for this question.)

\textit{HINT}

This is a critical point. If this is not obvious to you, think about the embedded
assumptions in the classic Black-Scholes-Merton financial option model,
regarding the actions of the holder of the option when it matures. That
involved (in the pristine world assumed for the purposes of the model) only
a simple decision to redeem, or let expire, the option. Now consider the
dramatically more complicated, expensive, and time consuming exercise of a real option such as the ability to shut down or start up a mining operation.

**SECTION 7.13. Review Questions for Chapter 13: Traditional Valuation Methods**

Consider the following questions:

1. **Income and other approaches**
   The income method appears to be the most arithmetically interesting of the three traditional approaches. In what situations would you *not* use this approach?
   Does the number of calculations, or their sophistication, indicate the accuracy or reliability of a valuation estimate?

2. **The dividend discount model**
   Consider again the dividend discount model in which value is determined by the discounted future dividends paid by a firm. How is the form of business organization (S-corp, C-corp, Partnership) important in its value, using this algorithm?
   This and the related questions about the dividend discount model could be adapted to a “net cash flow to equity” or “FCFE” model as well.

3. **Value and taxes under the dividend discount model**
   Continuing the prior question, what would be the value estimated using the classic dividend discount model for an S corp, C corp, and partnership that had the same operating profits, and whose owners had the same equity discount rate? How would income taxes at the corporate and investor level affect the calculation?

4. **Irrationality and value change if the corporate form changes**
   Assume, for the moment, that neither investors nor managers are irrational, and that investors and managers both seek to maximize income to investors. Assume that income taxes are levied at the corporate level for C corps, and that income taxes are also levied on dividends
and distributions. Does the dividend discount model tell us that the value of a firm changes based on form of organization? Is it rational for investors to invest in a C corp? Does the answer change if dividends are taxed at a lower rate, or if tax rates are different?

5. Free cash flow to the investor
   Consider the FCFE model, holding equity discount rates, taxes, and firm operating profits constant. How is this model different from the classic dividend discount model?

6. Free cash flow to the firm model
   Consider the FCFF model, again holding equity discount rates, taxes, and firm operating profits constant. What additional factors must be considered before we can conclude that the value under the FCFF and FCFE and dividend discount models all produce the same result?

7. Accounting Statements and Valuation
   Consider the following statement from a Wikipedia entry:
   One of the first techniques that a business valuation professional applies is called “normalization” of the subject company’s financial statements. Normalizing the company's financial statements permits the valuation expert to compare the subject company to other businesses in the same geographic area and industry, and to discover trends affecting the company over time. By comparing a company’s financial statements in different time periods, the valuation expert can view growth or decline in revenues or expenses, increases or decreases in assets or liabilities, or other financial trends within the subject company. Valuation professionals also review the subject company’s financial ratios, such as the current ratio, quick ratio, and other liquidity ratios; collection ratios; and other measures of a company’s financial performance.93

Given that Wikipedia entries are not definitive, do you agree with the “editor” of this statement?94

What, if any, adjustments you would make to “normalize” a company that had recurring expenses for the following: management training for a new executive team; start-up costs for a new office; bad debt

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94 Wikipedia entries may contain erroneous or incomplete information; change frequently; and reflect the whims or outright biases of the most recent “editor.” This makes it fertile ground for review questions—but not for authoritative statements.
losses higher than most companies in the industry; or unusually low entertainment expenses for a departing executive? Would your answer change if these occurred only once in a several-year period? What factors would you weigh in determining whether to adjust for these expenses?

8. Earnings capability, normalization, and value
Consider again the question above. Assume that your normalized statements for a profitable and well-established company included higher-than-actual earnings (because the normalizing adjustments eliminated extraordinary expenses in some years) for the past three years. Assume further that those normalized statements were provided to prospective buyers who otherwise had only the actual accounting statements.
Do you think the company would appear more or less valuable to those buyers?
Does your answer support the notion that valuation in practical terms is primarily a forward-looking exercise in estimating the ability to earn profits in the future; a backward-looking exercise in the evaluation of financial statements, which are assumed to reflect ability to earn profits in the future; some combination of both; or something completely different?

The “Synergy” Trap: M&A Variant
A merger is proposed between two large corporations that formerly competed against each other in at least one market. A financial analyst for an investment bank earnestly argues that the combined operation will achieve “synergies” that would result in hundreds of millions of dollars saved. The analyst points to costs of operations for each company currently, and subtracts out what she believes would be duplicative operations that could be combined. As a result, she values the combined company much higher than the sum of the market capitalizations of the separate companies.
Is this fundamentally an asset approach to valuation, or an income approach?
Why have so many such mergers failed? Is it the valuation method?
**HINT**

Why indeed have so many big mergers failed? Think about this in terms of the embedded assumptions in two valuation approaches:

- **Using the DCF model**, if you add up blocks of expected income from blocks of business operations, you are adopting the implicit assumption that these are additive without any additional costs or risks. Then you reduce costs for "synergies," now assuming that there are gains from coordination as well as efficiencies. The logic of the DCF model now causes the merged operations to appear to be worth more, and sometimes much more.

- **Compare the Value Functional approach.** Here the functional equation at the heart of the method is a mathematical structure that emerged from Control Theory. Thus, the ability to control, manage, maximize value, and otherwise coordinate activities is front and center. There is no embedded assumption that the same management (or different management), given twice as large a set of operations, would find the business twice as valuable.

Of course, the analyst using either of these approaches could force the result to reflect the naive "synergies" assumption mentioned above, but it is much easier to do so using the DCF approach. The inherent difficulties of coordination are front and center with the value functional approach.

9. **Internal Synergies**

A company with three divisions has a well-established value of $10 million. Revenue for each division, and internal profitability analyses, put the profit contribution of the two largest divisions at nearly $9 of the $10 million, even though revenue from all three divisions is about the same. Assume that the internal profitability analyses rely on cost allocations that are neither better nor worse than typical in similarly-sized companies.

The company wishes to sell off the smallest division, including all the assets that directly support it, and downsize slightly its corporate functions. As a first approximation, the firm’s internal accounting show that assets directly related to the smallest division account for $3 million of the $11 million in total assets, with accounting treatment not differing among the divisions in a manner that distorts the comparison.
How do you value the resulting firm, based on the information provided? Do the assets devoted to the smallest division matter, and if so, how and why?

10. Accounting information and asset methods
Consider again the preceding two scenarios.
What if the cost allocation methodology used by the company was faulty; would an asset method give the wrong answer because of it? How about an income method?

11. The “excess earnings” method
An old method of estimating the value of a privately-held business is known as the “excess earnings” method, because it based the value of a firm on the “excess” return above a minimum return on assets. Some authors have criticized the method and urged its abandonment; however, others still include it in valuation methodology guides.
What is your opinion on the method?
Do you think the intuition about “excess” earnings is substantially different from some modern portfolio theory models of stock market risk and return? If the intuition is similar, is it the execution, data, or something else that is wrong?


Consider the following questions:

1. Additivity of business values
You estimate that a shipping business is worth $1 million; you also estimate a packing business in the same city is worth $250,000. Can you now estimate how much they are both worth? Why, or why not?

2. Information needed to value dual businesses
Would your answer to the first question change if the two firm’s operations were located next door to each other, and they offered only
complementary services? If the current managers promise “synergies” exist between the two? If the investor considering buying the businesses knows a lot about packing, but nothing about shipping?

3. Expected value and gaming

The Low Stakes Casino offers a game that gives an expected value per dollar waged of $0.85, with a very modest betting requirement. The nearby High Stakes Casino offers a similar game with an expected value per dollar waged of $0.95, but with a very high betting requirement. The nearby movie theater offers a movie about gamblers. What is the best place to go for entertainment if you want to gamble, but can’t afford to lose too much? Did you consider not gambling in your answer? Is there a rational basis for gambling, or for purchasing equity in “gaming” companies?

4. Expected value and random variables: the mining firm example

You are forecasting the future earnings of a mining firm. You know that the revenue of the firm is heavily affected by raw material prices that are set on world markets, and are subject to considerable variation caused by weather and political disruptions. You also have access to several years of financial statements and price data, which show that the cost of mining stays relatively stable per unit over time, although the quantity demanded and price paid by customers does not. The firm expects to continue operating in the same manner as it has been in recent years.

If you are forecasting prices, quantities, expenses, and profits in the future, what variables do you consider “random,” either in a strict statistical sense or in a practical sense? For what variables would you use a statistical calculation such as “mean” or “variance”?

5. Cash-flow statements and “value”

An investor presents a detailed cash-flow schedule showing how the expected distributed income from a start-up company would, if discounted to the present time at 15% per year, be “worth” $1 million. The same investor presents a similar statement for an existing business, showing that the earnings for that company on an identical basis would be worth only half that amount. What reasons might an investor have to invest in the latter? The former? Neither?
6. \( E(NPV) \) and “value”

Numerous texts state that the “value” or “worth” of an asset is equal to the expected net present value of its future earnings. Why might this not be true? When would it be true?

**HINT**

*In a previous chapter, we suggested actually asking business owners about the worth of their company. If you have never asked an entrepreneur about his or her company, try doing that. Then ask yourself whether this individual would sell the firm he or she started for the expected net present value of its future earnings plus a dollar.*

7. Transaction costs and buy-sell transactions

Why do purchases and sales take place among informed investors that have access to the same accounting and historical financial information about publicly-traded firms? Do such parties, given nearly the same information, necessarily have the same expected net present value of the future earnings of these firms?

Does your answer change if you note that there are transaction costs (e.g., commissions, recording costs, legal fees) in most of these transactions, and that therefore the buyer actually spends more than the seller gets?

8. Profit maximization

Can you define the cost function for your operation (which might be a firm, a school, or a nonprofit)? Can you differentiate it? How about the revenue function?

9. Profit maximization

If defining, and then differentiating, the revenue and cost functions is so difficult, is profit maximization even possible for a firm?
HINT

This question requires a technical use of the term “maximization;” we are not asking whether managers just want to increase profits, we are asking if, at any single time period, they try to maximize them.

SECTION 7.15. Review Questions for Chapter 15: Value Functional Theory

Consider the following questions:

1. Extrapolation, “subjective” and “objective” forecasts

   Consider two value estimates done for a well-established firm with good accounting records:
   
   • The first is a careful calculation of forecasted earnings, with the revenue and expenses forecasted with an objective methodology of extrapolating the past 3 years of performance with any unusual (non-recurring) items removed, where the future expected earnings discounted using an internal cost-of-capital model.
   
   • The second is based on a subjective forecast of future earnings, based on a careful analysis of underlying market and business conditions, and a subjective assessment of how these conditions were likely to affect revenue and expenses, with expected future earnings discounted in the same manner as the first value estimate.

   Both estimates were prepared by individuals that were similarly knowledgeable about financial statements and the business. Which would you expect would be more reliable? What questions would you ask each analyst? What was the primary method that each used to create their estimate? Do you consider the objective trend forecast that ignores any expectations about future conditions, or the subjective forecast based on expected conditions, to be more reliable?

HINT

For a set of valuation estimates for a specific company, using different methods, including value functional, a reporting of analyst recommendations during the contemporaneous periods, and DCF with and without subjective adjustments, see Patrick L. Anderson, “The Value of an Expropriated Oil
2. Bellman Equation for a Mining Company

Consider a mining company, which owns natural resources and is attempting to maximize the benefits of running the mine over time. The company knows that market prices for the resources in the mine fluctuate, but tend to rise over time at a rate that roughly approximates inflation.

What would be the effect on the current-year production decision in the following market situations:

- Market prices go up, and you believe they are going up temporarily.
- Market prices are on a slow trend upward.
- Market prices are on a slow trend downward.
- Costs of production go up.
- Costs of carry (including the financing costs of holding the mine and operating the mining business) increase.

How would you write the reward function for a sample company, and what state variables would you use?

Compare the reward function with an accounting income statement. What are the action and control variables on the accounting statements? How is revenue for the firm treated in your illustration? Costs? Investments?

3. Real options

Suppose you are the manager of a company that has the option to purchase rights to exploit a mine over the next ten years, and the material from the mine is essential to your goods and services. Suppose the option is currently “out of the money” by a significant margin, that the materials produced in the mine are subject to significant price fluctuations, and you have the ability to rapidly scale up or down your production.
How does the $E(NPV)$ valuation approach treat your decision? How would a properly-defined value functional approach treat the same decision?

**SECTION 7.16. Review Questions for Chapter 16: Value Functional Applications**

Consider the following questions:

1. Write out a value functional equation, and specify the variables in that equation, for the value of a firm that sells food and drink on a beach that is visited seasonally by tourists. You can assume the owner has purchased or rented the rights to sell on the beach, and that people regularly visit the beach during the summer season, and that they are both hungry and thirsty when they do.

   **HINT**
   
   You can follow a simple logical progression here. Who are the potential customers? What do they want? When do they want it? What affects the answers to these questions.

2. Now do the same thing for an auto parts supplier that manufactures a set of parts for original equipment manufacturers (such as General Motors or Volkswagen) or one of their Tier I suppliers.

3. How does a real option to abandon, or substantially increase, an investment fit into the value functional equation?

   We provide additional information on the valuation of the example companies in Chapter 6, “Applications: Intermediate Results of Value Functional Examples in EBV.”

   **HINT**

   These types of real options are common in oil & gas, pharmaceuticals, and other firms where the ability to sell in the future is heavily dependent on having the rights to exploit inventions, reserves, patents, or other forms of
intellectual property. However, putting the real option into a value functional equation requires care. An example is the valuation of an oil & gas company with rights to shale oil reserves, described in Patrick L. Anderson, “The Value of an Expropriated Oil Company: The Case of YPF,” SI technical paper 2013-1 (March 2013); available from http://www.supportedintelligence.com. This technical paper demonstrates the use of both the value functional method and the traditional discounted cash flow method, and the significantly different results they can produce when real options are present.

### Section 7.17. Review Questions for Chapter 17: Applications in Finance and Valuation

Consider the following questions:

1. Distressed firms and valuation methods

   How does the degree of distress in a firm’s operations affect the valuation method selected? Consider how the apparent valuation of a distressed, but still operating, firm would be estimated using naïve asset, income, or market methods.\(^95\)

   **HINT**

   Think of this question first as if you were hired by the bankruptcy trustee (or another neutral party charged with managing the bankruptcy process).

   Now answer it as if you were a creditor (a party to whom the bankrupt firm owed money); a stockholder; a customer or supplier; and a state government concerned about loss of employment and tax revenue. Do you see how the various parties have different incentives? Do you see the rationale for governments to establish laws defining and regulating bankruptcy?

   Consider this advice again for question 5.

2. Distressed firm, assets, and equity

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\(^95\) By “naïve,” we mean using a method without considering whether or not the method is the best one to use, and without making necessary adjustments for factors other than those specified by the method itself.
How does distress affect the buyer’s decision about what to buy, when the buyer could choose the equity in a firm (including the ability to affect management and the dividend policy), or the assets of a firm (which may have more, or less, value outside of the firm)?

3. Bankruptcy and value
Why do some bankrupt firms “have value,” and in what sense specifically do bankrupt firms have some value?
Consider in your answer the difference between assets and equity; the value of brand reputation, supplier networks, and a customer base; and the changes that can be made under bankruptcy laws to business operations.

4. Liquidation and reorganization
Could some firms (or assets of such firms) have greater value in liquidation than reorganization?

**HINT**
The answer to this is clearly “yes.” Think about why that is the case even in theory, and when it is the case in practice.

5. Valuation method, bankruptcy outcome, and methods
Assume you are asked to estimate the value of a distressed firm that had substantial hard assets but had been losing money due partially to long-term contracts that specified a below-market price for its main product.
First, use naïve income and asset methods to estimate the value of the firm. Does the naïve value estimate change by method, or by bankruptcy outcome, or by both?

**HINT**
Read the hint for question 1.

6. Start up Firms
Consider the factors described in Chapter 4’s “Competing Theories: Agency, Scale, and Team” on page 52. Which of these could be assessed in a start-up firm? How would these affect the willingness of investors to support the nascent firms?

7. Distinguishing franchises from other forms of business
   What distinguishes McDonald’s (or Burger King, or Arby’s) from a local burger restaurant?
   Do any of these distinctions favor the local restaurant?
   What distinctions favor the franchised retailer?

8. Essential elements of a franchise
   Can you conceive of a franchise that had only two of the three essential elements?
   What would be the benefits to the franchisee and franchisor of such an arrangement? How could it be enforced across the entire franchise?

HINT

Think of the last time you stayed at a hotel, visited a fast-food restaurant, rented a car, bought a car, and shopped for clothes. Which of these were franchises? Do you know? Would the experience have been different if the answer to those questions were different?

9. Value of a franchise
   How would the value functional change for a firm that became a franchise?
   What risks, costs, and benefits, would necessarily be included?
   What new state and control variables would exist?

HINT

Think about the answer to question 8, from the perspective of the owner of the franchise.
10. Franchise infringement

How might a franchisor, or a franchisee, infringe on its agreement and reduce the business value of the other party? How might it reduce the value of other franchisees?

HINT

The author testified as an expert in a case in which the Court ruled that a franchisor of rent-a-car services violated the franchise agreement with its franchisees, with negative effects on the franchisee that served the State of Alaska. (The Court found that the franchisor had breached the agreement when it purchased a competing rent-a-car system out of bankruptcy, and then merged its sales, reservation, and marketing operations into those of the successful franchisor.) The jury in that case awarded $16 million in damages to the franchisee as a result. As reported in Bloomberg Business Week, the owner of the franchisee described the result as a victory for the small businessperson against a giant corporation, and as a victory for the franchise system in general:

On October 1, 2009, after two hours of deliberation, the jury awarded damages to the 50-year veteran franchisee [Robert Halcro] declaring:
We, the jury, based upon the preponderance of the evidence, unanimously find the following verdict on the damages suffered by Alaska Rent-A-Car Inc., as the proximate result of the breach by Avis Budget Group, Inc. and Avis Budget Car Rental, LLC in the following amount: $16 million.

In hearing the verdict, Halcro said, “We won big. This goes to the core of the American franchise system. Big franchisors shouldn’t be able to get away with what they are trying to do in going against the small franchise operator.” He said he felt fortunate that he had the money to battle in court, that most small business people don’t have the resources. “I spent almost $7 million fighting this franchisor; but I don’t care. I came into the world with nothing, my family is okay financially, so to hell with them. I’ll march on.”

The main facts of the case and the decision, as well as the arguments made by attorneys and experts for both sides, are now public record.97

Can you conceive of how the franchisee might be damaged if the franchisor does something that, in its opinion, will help grow its business across the country? What if that “something” involves undermining the competitive position of the franchisee? If this is interesting to you, read some of the

publicly-available documents for the case summarized above, including the arguments for both sides. Does your opinion change?

SECTION 7.18. Review Questions for Chapter 18: Law & Economics Applications

Consider the following questions:

1. Value of a company and lost profits
   Could the value of a company be less than the net present value of lost profits caused by a business interruption?
   Consider this question for start-up firms; large firms with substantial tenure in the marketplace; and small firms.

   HINT
   This is a recurring question within the community of forensic economists and forensic accountants that typically rely on the DCF method as the basic model for calculating economic damages. Note how the embedded assumption in the DCF model pushes the answer to this question in one direction. Now consider the different embedded assumption (regarding controlling the enterprise to maximize value) in the value functional approach. See also the hint for Chapter 13 regarding the value of merged companies and the episodes in financial markets of exaggerating “synergies” from mergers.

2. Methods
   Consider the traditional discounted cash flow methodology. How can it be adapted to handle mitigation?

3. Information necessary to value a firm

The publicly-available documents include the decisions in the trial court and the court of appeals: Alaska Rent-A-Car v Avis-Budget Group, (Federal DC for Alaska, No. 303-cv-00029, decided Sept. 2009; affirmed at 9th Circuit Court of Appeals, No. 10-35137, March 2013). Decisions in major cases in Federal courts (and some state courts) in the United States are often available from websites such as FindLaw or Justia, and in many cases from the court websites (often with registration required, as with the Federal courts’ Pacer system).
Think of the information necessary to value a firm. What additional information is needed, if any, for a damages estimate? Is less information needed for a damages estimate (for a significantly damaging event) than for a business valuation?

4. Best methods
What traditional methods would you use for the following damages cases:
• Breach of contract that shut down the one retail store owned by the firm for 6 months
• The same event, shutting down the retail store permanently.
• The same event, when the store is owned by a national chain.
Would you consider using a real options, or value functional approach?

5. Policy risk and employment
Review again the example of policy risk and unemployment. (You may want to review also the discussion of this topic elsewhere in this Solutions Manual.) Now consider the changes in policy that took place in the years 2011-2013 as they affected employers in the United States, including:
• Increases in unemployment insurance costs, due to the extension of unemployment benefits encouraged by federal policy;
• Mandated payment of health insurance for most employees, or payment of a fine (tax);
• Increased income taxes on many owners of pass-through firms (LLCs, chapter S companies, partnerships) who report their share of company earnings as personal taxable income.
Do you think these policy changes affect the income or incentives facing employers?
Do you agree with the intuition that a manager’s strongest motivation is usually not to maximize profits, but to maximize value?
Do you agree with the intuition that increased policy risk could discourage some hiring, even if it didn’t immediately reduce profits?
HINT

As of the beginning of 2013, business owners in the United States could anticipate the following policy-related changes that would directly affect their costs of employing workers over the next year and a half, varying by a host of factors including state and by size of company: the implementation of “Obamacare” in the beginning of 2014; the implementation of state health care exchanges (or reliance on federal exchanges) at that time; “solvency taxes” on unemployment insurance in many states; high unemployment insurance rates due to both persistent high unemployment and policy-related extended benefits; new taxes on “investment” income; higher payroll taxes; new taxes on income (including pass-through income from S corps, LLCs, partnerships, and sole proprietorships) for those whose combined earnings (including both wage and business earnings) are above approximately $250,000 per year; beginning in 2013; fines for employers of more than 50 people that do not provide health insurance; and compliance with extensive regulations regarding disclosure of income data to federal agencies to allow them to enforce new laws requiring some employers to offer health insurance to some workers depending on their household income. This does not include other proposed changes, including an increase in the federal minimum wage, a renewed extension of unemployment benefits, and changes in the tax treatment of corporations doing business in multiple countries. It also does not include the penalty (a tax, according to the Supreme Court’s ruling) on individuals that do not purchase health insurance, nor the assessment on health insurance policies, nor the regulations on the type of coverage that must be included in those policies.

Do you think this represents significant uncertainty? Do you think it affects the motivation to hire, and the resulting employment in the general economy?

Regardless of your economic assessment above, do you think that a person’s view of the degree of uncertainty, and the burden it places on employers, tends to change with his or her political views (within the US political spectrum)?
Appendix A

Guide to Mathematics of Value

by Patrick L. Anderson,

with contributions by Manav Garg

An Introduction

This Appendix describes a set of algorithms that are useful in the study of the value of financial assets or businesses. An algorithm is a defined procedure for solving a problem using specific inputs. This topic is sometimes incorrectly shortened to “formulas,” but the term algorithm emphasizes the fact that, without getting the right inputs, and using the correct “formula,” the result is likely to be useless.

This section outlines powerful concepts used in economics and other social sciences. In particular, we introduce the prevalence of random variables, the concept of a measure; the idea of a distribution of random variables, and the use of the expectation statistic to condense our understanding of an important aspect of the distribution of random variables.

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1 In mathematics and computer programming, algorithms are often the key building blocks for performing useful work. The word is based on the name of the Islamic mathematician and astronomer Al-Khowarizmi, who served in the court of Caliph Al-Mumun in Baghdad circa 820 A.D. The term “algebra” is also attributed to his written discourse on mathematics that was later translated into Western European languages. Apparently, the arabic term “Al-jabr” meant both the mending of broken bones, and the solutions of mathematical equations—an historical precedent that may seem ironic to many a student.

A formula may be at the heart of an algorithm, but is meaningless without the information on the inputs and the operations that are embedded in a formula.

For example, the “formula” for “net present value” involves a number of arithmetic operations (addition, division), as well as specific concepts (time, discount rate, cash flows). Formulas can be very useful when they summarize the calculation portion of an algorithm for individuals that share knowledge of the inputs and operations involved. Formulas used without such shared knowledge can produce results that are useless.
We also describe in the appendices certain essentials of optimization. This includes the dynamic programming technique that involves a value functional equation, which is commonly used in control theory, and elements of calculus.

SECTION A.19. Random Variables, Measure Theory

a. Random Variables

A “variable” is a concept that can take on different arithmetic values. In mathematical finance, variables such as $d$, $r$, and $WACC$ might be used for the discount rate, the interest rate, and the weighted average cost of capital, respectively. In each case, the variable could take on one value in one case, and a different value in another case.

Random variables are variables that are dependent on some random event; an equivalent term is stochastic. The outcome of a stochastic or random event is determined by some process which cannot be completely predicted.

If we know the likelihood of events occurring due to a specific random process, then we know its distribution. Of course, one can completely understand the likely distribution of random events, but not know what particular event will occur until it happens. A flip of a fair coin is an example of this.

Random Variables in Business

Almost all important concepts in business involve random variables. The sales revenue in the current quarter, or month, or year is almost always random. Even for widely-followed, publicly-traded companies, sales vary from quarter to quarter, and not always in the manner expected by management or investors. Most items in the accounting income statement, and the accounting cash flow statement, are random variables.

It is important to note here that historical costs are not random events after they have occurred. After the transaction occurs, it is a historical fact. Thus, most items in an accounting balance sheet are based on historical events, and are not random variables at the time they are
prepared and distributed. However, forecasts of those same variables will almost always involve considerable uncertainty, and that means that forecasts are fundamentally random variables.\textsuperscript{2}

Modeling the finances of a company will involve dealing with both random and deterministic variables.

\textbf{b. Probability Space, and Measure}

To work capably with random variables in finance, we need to carefully define a stochastic process, and the fundamental principles of probability.

To do this, we first need to define a \textit{probability space}, which includes three elements: a \textit{sample space} full of possible outcomes or \textit{states of nature}; a set of \textit{events}; and a probability \textit{measure}.\textsuperscript{3}

1. The sample space $\Omega$, which is a finite set.
   All possible events are within the sample space, including the null set.

2. A collection of subsets of $\Omega$ on which probability is defined, which we call the set of events $F$. (This may also be described technically as a \textit{tribe}, \textit{algebra}, or \textit{field}.) The null set is also a member of $F$.
   This set of events follows certain rules of logic that are easily stated in set theory. In particular, if two events are in the set, then their union are also in the set of events; and if one event is in the set, then its complement is also in the set.

3. A probability measure $P$, which is a non-negative set function defined on $\Omega$.
   Probability measures follow specific rules. These include the requirements that the measure of any specific event must be between zero and 1, and that the measure of all possible events must sum to exactly one. This has a clear intuitive meaning: the probability of an event cannot be less than zero, the probability of a few of the possible events must be more than zero, and the chances of all events must be 100%.

The probability space is sometimes designated as a \textit{3-tuple} or “triple” of symbols $(\Omega, F, P)$ each denoting one of the elements.
Examples of Probability Spaces

Consider the following example probability space for a farmer (agribusiness seller) or a purchaser of farm goods.

- The states of nature are the possible states that could occur.
  In this example, we might have “good” weather or “bad” weather next season; we might have “high” or “low” rainfall; and prices for fuel might be high or low.
- The set of events describes the instances where something random will occur.
  These events, in this example, would include the weather, the price of fuel, and the rainfall.
- A measure of the events.
  We might use inches of rainfall, dollar prices, or some indicator of weather.

Probability Axioms

The probability measure is the tool we use to measure probability. Such a measure must obey the probability axioms, such as the axiom that all probabilities must sum to one, and that the probability of every event must be a number between zero and one, inclusive. Although colloquial discussions of probability usually use one of a small handful of probability measures, in fact there are a number of possible measures.4

Random Variables: A Precise Definition

Given a probability space \((\Omega, F, P)\), we will consider a random variable, a function \(X\) (from the states of nature to the set of real numbers) that is \(F\)-measurable, meaning that we can measure the probability of events within our probability space.

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4 Probably the most common measures are the “frequency” measure, in which the probabilities are based on the frequency of the events in the past; and the subjective measure, in which the probabilities are assigned solely on the subjective assessment of the individual. In this latter measure—quite common in sports betting—individuals usually disagree on the likelihood of future events. Indeed, the industry of sports betting depends on disagreements among participants about the likelihood of events!
c. Measure Theory References


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**SECTION A.20. Stochastic Processes**

A *stochastic process* (or random process) within the probability space is a collection of $X$-valued random variables indexed by a set $T$ (“time”). This is sometimes called a *filtration* of the set $F$. Logically, the set of numbers in the filtration that has a smaller time index is a subset of one that has a higher time index. This has a natural meaning in terms of the information available to an investor; if the filtration of information on prices as of the current time is denoted $X_t$ then the information contained in $X_{t+1}$ will naturally be greater.

**d. Markov Processes and Martingales**

Two important properties of certain stochastic processes that arise in mathematical finance are the Markov property and the Martingale property. These properties are not identical, despite the alliteration and occasional confusion in use.

A Markov process is a stochastic process in which the available information about the probability distribution of the next period’s outcome is entirely captured in the last period’s value. Many economic and business variables appear to behave like Markov processes, in that the current level of the variable is by far the most useful indicator of the next period’s value. For this reason Markov processes are sometimes called “memoryless.”
In contrast, some variables are “path dependent,” meaning that one can usefully observe the path a variable has taken when predicting the next value. Variables that tend to follow a trend are examples of these, as are some exotic options.

A “Markov chain” is a Markov process with a finite state space, where the transition probabilities that govern movements from one value to the next are often denoted by a transition matrix $P$:

\[
P = \begin{bmatrix} p_{ij} \end{bmatrix} = \text{transition matrix; where } \]
\[
p_{ij} = \Pr \{ x_{t+1} = j \mid x_t = i \}, \text{ at time } t; \text{ and } i, j \in S = \text{finite state space.}
\]

If the transition probabilities do not change over time, the Markov chain is stationary, and has a steady-state distribution that is determined by its transition matrix.

**Martingales**

The related concept of a martingale is a stochastic process in which the expected value of the next observation is equal to the last observation:
Martingale

(EQ A-3)

\[
E(|x_n|) < \infty; \\
E(x_{n+1} | X_1, \ldots, X_n) = x_n.
\]

Apparently, the term “martingale” originated from a betting strategy practiced in 18th century France.

Note on “equivalent martingale measure”

We use the term “equivalent martingale measure” in a specific way that is equivalent to the “risk neutral” measure. Note that any expectation is dependent on a probability measure, and hence the martingale property must be considered under specific measures.

SECTION A.21. Brownian Motion; Itô Processes; Geometric Brownian Motion

a. Brownian Motion

A very useful model of the behavior of business and natural phenomena over time is known as Geometric Brownian Motion or “GBM.” It is the basis for the most fundamental financial option-pricing models, but has much broader applications for asset prices, revenue growth, etc.

Standard Brownian Motion is a stochastic process commonly known as a “random walk,” in which each increment (“step” in the “walk”) can be up or down. It has a colorful history beginning with botanist Robert Browning and extending through Louis Bachelier, Albert Einstein, and Norbert Wiener. Sample paths of a true random walk look jagged, unpredictable, and without any trend—which is exactly what Brownian motion is. The properties of standard Brownian motion are summarized in equation (EQ A-4).
Louis Bachelier proposed Brownian motion as a model for stock prices in his now celebrated (and long neglected) 1900 monograph. However, a simple random walk is not sufficient, as stock prices have a noticeable trend.

b. Geometric Brownian Motion and Itô Processes

Geometric Brownian Motion or “GBM” is the most common model for the process followed by stock prices over time, and was the basis for the Black-Scholes-Merton formula for financial options. In a GBM model, there is an underlying “drift” to the process that is proportional to the stock price, along with a random component that is similarly proportional to the stock price. This matches the intuition that investors consider changes in value (“returns”) due to underlying growth over time and variations due to random events (“risk”) largely in terms of the fraction of the amount invested.

Geometric Brownian Motion is part of a larger class of Itô Processes, which have both a “drift” and “diffusion” component, with the latter driven by a standard Brownian motion. These are described in equation (EQ A-5), which includes both the formal and informal notation for the process.

Illustration: the Informal GBM Equation

The formal notation illustrates that the value at time \( t+1 \) is an increment different from the value at time \( t \). The informal notation focuses just on this increment, using (with some abuse) the notation common in calculus.

In this notation, the process \( dS/S \) appears to be the fractional change in value of the underlying stock, and the “instantaneous drift” and the “instantaneous standard deviation,” appear to be the \( \mu \) and \( \sigma \) parameters in the formal GBM equation. This has a strong intuitive basis; for
example, a gain of $2000 on a $10,000 equity investment during a year could be understood as \( \frac{dS}{S} = 20\% \), which is equal to the sum of an underlying drift \( \mu = 15\% \) and a random \( \sigma = 5\% \) upward movement. This is a direct, though not precise, interpretation of the informal GBM equation, with \( dt = 1 \text{ year} \) and \( dz = +1 \).\(^5\) If, continuing the example, the following year the gain on the same stock investment was 10%, it could be interpreted as the same drift and diffusion process continuing, only this time with the random \( dz = -1 \).

However, \( dS \) is not a random variable, so it does not actually have a mean or variance. The precise interpretation of the elements of the differential version of the Itô equation are described in references such as Duffie (2001, section 5C).

---

**Itô Processes and GBM**

\[
S_T = S_0 + \int_0^T \mu_s \, ds + \int_0^T \sigma_s \, dz; \quad \text{where} \quad \int_0^T |\mu_s| \, ds < \infty, \quad \text{all} \ t
\]  
(Ito process)

\[
S_T = S_0 + \int_0^T \mu S_t \, dt + \int_0^T \sigma S_t \, dz
\]  
(Ito process (GBM); formal integral notation)

\[
\frac{dS}{S} = \mu dt + \sigma dz
\]  
(GBM; informal notation)

---

c. Properties of GBM

The natural properties of GBM for modeling equity prices, and the subtle contrast with bond prices, are illustrated by the properties shown below. GBM is known as a log-normal process because it produces returns that are normally distributed, meaning that the natural log of prices are normally distributed.

An often-overlooked result of the underlying Wiener process is visible from comparing the expected value of a GBM process with an underlying drift parameter \( \alpha \) and the risk-free bond with an underlying drift rate of \( r \). Note that the expected value (and the “instantaneous drift” of the informal GBM equation) is *not* the same as \( \alpha \). Instead, it is composed

\(^5\) Those more familiar with Itô processes might note that the time increment here of one is conveniently chosen so that the square root of the increment of \( t \) is also one.
of the underlying drift and half of the instantaneous “variance.” By comparison, there is no variability in the price process of the (risk-free) bond in our example, and the drift parameter \( r \) does not need to be modified for any risk when calculating an expected value.

This leads to a critical point: to the extent actual processes important to businesses operate in a manner similar to geometric Brownian motion, the volatility as well as the trend growth rate directly affects the future expected value.

\[
S_t = x \cdot \exp\left[\alpha t + \sigma z(t)\right]; \ x = S_0 > 0; \ \alpha, \ \sigma \ \text{constants}
\]

\[
dS_t = \mu S_t dt + \sigma S_t dz(t); \ x = S_0;
\]

\[
\mu = (\alpha + \sigma^2 / 2)
\]

\[
E(S_t) = x \cdot \exp(\alpha t) \cdot E\left[\exp(\sigma z(t))\right]
\]

\[
= S_0 \exp\left[\left(\alpha + \sigma^2 / 2\right) t\right]
\]

\[
B_T = B_0 \cdot e^{rT} \quad \text{future value (at time } T) \text{ of bond with price (at time zero) of } B_0, \text{ assuming no default risk and constant interest rate } r
\]

\[
dB_t = rB_t dt
\]

---

**SECTION A.22. Change of Measure**

**a. Why Change the Measure?**

The concept of *change of measure* is used in mathematical finance, especially the derivation of the “risk neutral” or “equivalent martingale”
pricing formulas. To determine one measure (such as a “risk neutral” measure) from another observed probability measure, we will need a change of measure. The existence of such a random variable is based on a fundamental theorem from measure theory known as the Radon-Nikodym theorem.

b. Equivalent Probability Measures

Starting from the original probability space \((\Omega, F, P)\), we know that the measure \(P\) of the probability of events will follow the probability axioms.

Suppose there is another probability measure, following the same probability axioms as \(P\), that is available on the same sample space and events. If the second measure assigns a zero probability to the same set of events as the first, we call the two measures equivalent. Note that this does not mean the probabilities for specific events are equal; in fact, unless they are different the entire idea loses meaning. Equivalent probability measures both sum to one, and assign zero probability to the exact same set of events. However, within those events, the specific fractions are different.

\textit{Change of Measure}\footnote{The change of measure is presented in a number of references. These include the compact presentation in Duffie (2001, appendix A), and the more complete explanation in Shreve (2004, vol. II, section 1.6) or Shreve (2004, vol. I, section 3.2).}

Suppose Poonam and Quincy go to the horse races. Poonam carefully studies the \textit{Gazette} and assigns odds to each horse. Being a good horse gambler, she of course knows that the “odds” have to sum to one. On the other hand, Quincy just picks horses on the basis of their names. Although he is cavalier about picking his favorites, he is quite rigorous about making sure that his probabilities sum to one. Both ignore any horse that is not in the race, for obvious reasons.

These pair of subjective probability measures—call them \(P\) and \(Q\)—are equivalent.

We would like to define a change of measure that translates the probability under one measure to the probability on the other. This would allow us, for example, to determine Quincy’s odds from Poonam’s. We outline below how this could be done.
The probability of the individual events are, under the measures $P$ and $Q$, denoted by lower-case letters such as $p_1$ and $q_1$ for the first event. Of course, the sums of all these (nonnegative) fractions $p$ and $q$ are one for each measure. Using the horse-racing example, Poonam and Quincy assign different probabilities to each horse, such as $p=.25$ while $q=.30$ for the first horse.

However, using a change of measure, we can express the one in terms of the other. We define a non-negative random variable $Z$ on $\Omega$ according to the following rule:

\[
Z(\omega_i) = \frac{q_i}{p_i}, \quad i = 1, 2, 3
\]

Change of Measure $Z$ \hspace{1cm} (EQ A-7)

Recall that both bettors ignore horses that are not in the race and consider the probability of some horse winning to equal 100%. This is the key part of the definition of an equivalent probability measure, and forces the fractions in the equation above to be positive rational numbers. Thus, the $Z$ variables provide a change of measure, allowing us to know the odds Quincy places on a horse simply by observing the odds that Poonam places on the same horse.

c. The Radon-Nikodym Theorem

Now, according to the Radon-Nikodym theorem, under conditions that require a finite and continuous measure function, there exists a unique random variable $Z=dQ/dP$. In the horse-racing example, the Radon-Nikodym derivative is simply the quotient of two probability measures of a specific event.

Extending this insight to expectations, consider the random variable $X$ for which both $P$ and $Q$ are equivalent measures. The probability distributions of $X$ are different under the two measures, which means the expectation (and potentially other statistics) are different as well. This change of measure variable $Z$ has a very useful property when calculating the expectation of a random variable $X$: 
Expectation Under Change-of-Measure (EQ A-8)

\[ E^Q = E^P (ZX); \]
\[ Q, P = \text{probability measures}; \]
\[ Z = \frac{dQ}{dP} = \text{Radon-Nikodym derivative.} \]

The expectation under the one measure is the same as the expectation of the random variable multiplied by the change-of-measure variable, with the later expectation taken under the other measure.

**Stochastic Discount Factor (Pricing Kernel)**

The stochastic discount factor, or pricing kernel, is an application of the Radon-Nikodym theorem in the context of a continuous-time stochastic process. Recall that the SDF can be expressed as the product of a gross discount rate and a change-of-measure for the actual and “risk neutral” probabilities:

SDF and R-N Derivative (EQ A-9)

\[
\phi_i = \frac{q_i}{\pi_i} = \left(\frac{1}{1 + r}\right)^{\pi_j} = \text{pricing kernel} = \text{SDF.}
\]
\[
\phi_i = \frac{\left(\frac{1}{1 + r}\right)^{\pi_j} \pi_j}{\pi_j} = R^{-1} \frac{\pi_j^*}{\pi_j};
\]
\[
\frac{\pi_j^*}{\pi_j} = \text{change of measure.}
\]

The change of measure here is a Radon-Nikodym derivative.
SECTION A.23. Expectation of Random Variables

In general, if \( X \) is a random variable defined on a probability space, then the expected value of \( X \), (denoted \( E(X) \)) is defined as

\[
E(X) = \int_{\Omega} X dP
\]  

(EQ A-10)

where the differential \( dP \) refers to the probability measure in the probability space \( \Omega \), and the indication of the state space indicates the use of the Lebesgue integral. This technical definition covers situations that are unusual in practical settings.

A more typical setting involves a random variable \( X \), its realizations \( x \), and its probability density function \( f(x) \). The density function produces a non-negative number of one or less:

\[
P \{ X \in B \} = \int_{B} f(x) dx;
\]

\[
P \{ a \leq X \leq b \} = \int_{a}^{b} f(x) dx;
\]

where:

\( B \subseteq \mathbb{R} \) \text{ [ } B \text{ is a subset of Real Numbers]}

Although the density function is not technically a probability measure, it does have a natural interpretation as the number of times that a specific event occurs within a particular interval. This is clearer in the related concept of a probability mass function, which (in discrete populations) is defined as the probability of a specific realization of a variable.

Where a density function is defined for a random variable, the familiar Reimann Integral can be used to calculate the expectation of the variable.
The simplest case of this is when there are a limited number of discrete potential outcomes and a known distribution. For example, consider the expected value of a fair die, which has six sides numbered from one to six. The expected value of the roll of the die (the “event” is the roll, the “outcome” is the number showing on the die after the roll) would be the sum of all six numbers on the die, each multiplied by 1/6. (We calculate this below.)

**d. Aspects of Expectation**

The expectation operator has important properties worth noting:

1. **Expectations admit a linear transformation**

Note the expectation operator is not invariant to other transformations. In particular, $E[g(X)]$ is not the same as $g[E(X)]$. One transformation of note under which expectation is not invariant is squaring the variable; see the discussion of variance and second moment.

2. **Expectations of time-series variables are taken at specific times**

Expectations in finance typically involve stochastic processes that evolve over time. Such expectations must be defined in terms of the
information available at the time of the expectation. For example, the “expected value” of an election outcome, for example, changes markedly as we near, and then pass, and election date. We define expectation at time $t$ as the expectation conditional on all the information available at time $t$:

$$E_t(Y) = E(Y | F(t)) = E(Y | X(s), 0 \leq s \leq t).$$

Correct notation reminds us of the time period at which the expectation is taken.

3. The Law of Iterated Expectations

An expectation taken with respect to information at a later date will include all the information available at an earlier date. Looking again at equation (EQ A-14), we can observe that the expectation denoted $E_t$ is conditional on the information set $F(t)$. Intuitively, if we again took an expectation (after using the information set available at $t$), but made it conditional on the information available at an earlier time, such as $F(t-1)$, we would add nothing to our knowledge. Conversely, if we are limited to one set of knowledge, obtaining and then discarding more does not improve our results.

A formalization of this is known as the Law of Iterated Expectations. A conversations version, due to Cochrane (2003, section 8.1), is the following: “your best forecast today of your best forecast today is the same as your best forecast today.”

4. The conditional expectation of random variables is a random variable

An expectation implies that the variables are random. If all uncertainty is resolved with an expectation—meaning that the expectation is unconditional—then the resulting quantity may be fixed. However, in most cases involving business and finance, the actual expectation is taken conditional on an information set that includes random variables. In such cases, the (conditional on a random variable) expectation is also a random variable.
There is an obvious implication of this for business valuation: given that almost all business revenue is affected by economic, industry, and other conditions that are subject to uncertainty, the expected net present value of a stream of future business earnings is almost always a random variable.

5. Expectation Based on Probability Measure

Remember that the probability for the expectation algorithm is one based on a specific measure. If we change measures, we also change the expected value. This is the basis of the “change of measure” theorem.

e. Limitations of Expected Value

The expected value is a powerful concept, and one that we will use repeatedly. However, it has significant limitations that will become more apparent when we discuss risk. These include:

1. The expectation captures—extremely well—only one dimension of the distribution of a random variable: its mean. In finance, it is easy to be right about the mean of important variables—but make disastrously wrong business decisions based on that information.8

2. Other measures of central tendency such as the median are more robust to outliers. A single unusual observation in a sample (or a single incorrectly-typed number) can significantly affect the sample mean, but may not affect the median at all.

3. Another key dimension of the distribution—its variance—is not captured by expected value. The expected value of a very widely-distrib-
uted variable can be exactly the same as one for which there is only a narrow distribution. We discuss this in b., “Measures of Variation: the Variance.”

4. Although sometimes overlooked, most variables in real life do not have the neat symmetrical distribution epitomized by the “bell curve” of a normal distribution. Most random variables in practical use are skewed in one direction or another. The expected value does not capture this.

**Section A.24. Statistics of Central Tendency and Variation**

The mean is only one of multiple statistics of central tendency. We will consider multiple statistics for central tendency and variation in this section.

**a. Median and Mode**

The *median* is another common statistic; the median is the point at which an equal number of observations have higher and lower values. In nice, symmetric distributions (such as the classical “bell curve”) the median and the mean are identical. In most actual distributions in business economics, the median is not the same as the mean.

The *mode* is the value that appears most often. Again, in the classical bell curve distribution, the mode equals the mean equals the median.

**b. Measures of Variation: the Variance**

The expected value, which is often the mean of a distribution, is usually the first indicator of interest about a variable. People, whether they are investors or not, typically want to know “on average” what something costs, or how long an event takes, or how big a pet is likely to grow. For this reason, calling it the “first moment” makes intuitive sense; we discuss below the “method of movements” and this nomenclature.
However, in finance and valuation, we will almost always be interested in the second moment about the mean, which is a measure of variation. We will often use the variance statistic to measure this variation.

The variance is defined as:

\[
\text{Variance} \quad (\text{EQ A-15})
\]

\[
\text{Var}(X) = E[(X - E(X))^2]
\]

Note that the variance formula is the average of the squared deviations. This ensures that all deviations are counted as positives, and that the implicit penalty grows with the square of the distance from the mean. The variance is also the second moment about the mean, which is discussed in Section A.25., “Moments and Other Measures.”

**c. Variance and Covariance**

A similar metric to the variance of one random variable is the covariance between two:

\[
\text{Covariance} \quad (\text{EQ A-16})
\]

\[
\text{Cov}(X, Y) = E[(X - E(X))(Y - E(Y))].
\]

Note that the covariance of two independent variables is zero. The covariance measures the linear relationship among two random variables. The covariance between two variables, scaled by their variances, is the correlation:
Correlation and Covariance (EQ A-17)

\[
\rho_{X,Y} = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{E((X - \mu_X)(Y - \mu_Y))}{\sigma_X \sigma_Y};
\]

where:

\[
\mu_X = E(X), \mu_Y = E(Y);
\]

\[
\sigma_X = \sqrt{\sigma_X^2} = \text{Standard Deviation of } X.
\]

\[-1 \leq \rho_{X,Y} \leq 1.\]

Note that the correlation cannot exceed one in value, and that a correlation coefficient near zero indicates a very small linear relationship between two random variables.

**Correlation and Causality**

A common misconception, encouraged by popular sayings (such as “where there’s smoke, there’s fire”) advertising, political rhetoric, and advocates in criminal prosecutions, is that correlation implies causality. In mathematical terms, this is clearly not the case; very good advice in social science is to graph the relationship between key variables rather than assume a simple statistical relationship (such as correlation) adequately summarizes it.

**SECTION A.25. Moments and Other Measures**

We mentioned above the old tradition in statistics of describing distributions in terms of moments. Moments in statistics grew from an even older tradition of moments in physics. We will define this concept now, and place the mean and variance within the class of statistics known as moments about the mean, and moments about zero.

**a. Raw Moments**

Intuitively, a raw moment is the probability-weighted sum of the differences between a set of observations and a specific reference number.
When that reference number is zero, we get an indication of the values of the numbers in the distribution. When that reference number is the mean, we get an indication of how the values vary.

b. First moment about zero: mean

The mean is the first moment about zero. Intuitively, we can see that taking the differences between the observations and zero, and then weighting those differences by the probability or frequency of the observation occurring, gives an expected value. That expected value is the central point in the distribution.

c. Moments About the Mean

Moments about this central point are known as “central moments” or “moments about the mean.” The \( n \)th central moment of the probability distribution of the random variable \( X \) is defined as:

\[
\mu_n = E[(X - E(X))^n]
\]

Moments can also be normalized by dividing the raw moment by a factor based on the standard deviation of the variable. Such a standardized moment is a dimensionless number and can be compared to those calculated for variables of different scales.

d. Second central moment: variance

The second central moment is the variance. Intuitively, the variance sums up the square (second powers) of the differences between the observations and their central point. This sum of squares cannot be negative, as the distribution always extends outward from its central point and the distance from one point to another cannot be less than zero. The formula for the variance was given in Section A.24., “Statistics of Central Tendency and Variation.”
e. Third central moment: skew

The third central moment is a measure of the “leaning” or the “lopsidedness” of the distribution. Most distributions in practice “lean” one way or another; that lean may be confined to small portions of the distributions that are distant from the central point (often called the “tails”), or may characterize the entire distribution.

The skew statistic is the normalized third moment about the mean. Because the third moment is based on differences raised to the third power, it is possible to have negative values. A symmetric distribution will have a zero skewness; one that is lopsided to the left will have a negative skewness.

\[
\text{Skew} = \frac{\mu_3}{\sigma^3} = \frac{E[(X - E(X))^3]}{E[(X - E(X))^2]^{3/2}}
\]

f. Fourth central moment: kurtosis

The kurtosis statistic indicates whether the distribution appears short and spread out, or thin and tall. The kurtosis statistic is based on the normalized fourth moment about the mean:\(^9\)

\[
\text{Kurtosis} = \frac{\mu_4}{\sigma^4} = \frac{E[(X - E(X))^4]}{E[(X - E(X))^2]^2}
\]

g. A note on “methods of moments”

This tradition in statistics is the basis for one of the oldest statistical methods, the use of “ordinary least squares” (“OLS” or just “least squares”) method in regression equations.\(^{10}\) This method minimizes the squared deviations from the estimated value, which is equivalent to min-

---

\(^9\) Some authorities define the kurtosis statistic as the fourth moment about the mean, less 3. This makes the kurtosis of the normal distribution equal to zero.
imizing the second moment about that value. When the data and model actually fulfill the assumptions used to derive the classic OLS model (including availability of data observed without errors, normally distributed errors, lack of serial correlation, and a known linear relationship among the variables), the least squares estimators are the best linear unbiased estimators available.\textsuperscript{11}

An extension of this method is the Generalized Method of Moments ("GMM") estimating technique, which was developed recently and has been used in finance and other areas of economics.\textsuperscript{12}

\textbf{SECTION A.26. Optimization Using Differential Calculus}

Optimization is not a single algorithm; we outline here optimization using differential Calculus. Because Calculus provides a direct route to discovering extrema for continuous functions, it also provides a powerful tool for optimizing a certain quantity, when that quantity is determined by a function.

Extensions of the simple optimization presented here will be used in maximization across states; dynamic programming and the Bellman equation; and numerous other topics in finance and economics.

\textbf{a. Optimization and the Calculus}

Consider a function $y=f(x)$. Here the function is denoted $f$, the argument to the function is denoted $x$, and the output of the function is $y$. At the

\footnotesize

\textsuperscript{10} The common term “regression” dates from Sir Francis Galton’s 19th century study of the height of children (and also the size of plant seeds), in particular Galton (1886) “Regression towards Mediocrity in Hereditary Stature,” \textit{Journal of the Anthropological Institute} 15: 246–263.

\textsuperscript{11} Many statistics and econometrics books outline the least squares estimators and their properties. Kmenta (1997) is a favorite of mine for its clarity, Pindyck & Rubinfeld (1997) for applications. More recent textbooks in common use include Greene (2007) and Woldbridge (2001), (citing the most recent editions.)

\textsuperscript{12} Cochrane (2001, part II) contains a brief outline of the technique and numerous applications in finance. The GMM method is often attributed to the American economist Lars Peter Hansen, with the original references being Hansen (1982) or Hansen & Singleton (1982).
introductory level, and often in abstract models, these are considered single variables (scalars). However, in most business applications, the arguments to a practical function are multiple individual variables. These may be considered a vector and the function a vector-valued function.

If the function is continuous and differentiable, then the function will have a derivative at each point. The derivative indicates the rate of change of the function at a specific point. The first derivative is usually visualized as the slope of a tangent line to the graph of the function. It can also be visualized as the change in the function’s output value given a small change in the input value.

If \( y = f(x) \) is a function, the first derivative of that function can be denoted \( f'(x) \) or \( \frac{dy}{dx} \).\(^{13}\)

### b. Finding Extrema

The extrema of a function are the smallest and the largest values of the function. These extrema are often called min or max.\(^{14}\) For many simple linear functions (such as \( y = x^2 \) or \( y = -5 + 100x \)), the highest output value is produced by giving the function the highest input argument. However, for many others, including some “simple” functions like \( \sin(x) \), the answer is not so simple.

Extrema can be defined as local if they are the highest or lowest in a neighborhood or a range of numbers. An extremum can also be defined as absolute if it is the highest or lowest value over the entire domain of the function. This may occur between, or at, the endpoints. The criteria for a maximum of a function \( f(x) \), which could be a vector-valued function, is the following:

\(^{13}\) It is interesting to observe one of the most famous intellectual battles in Science, over the originator of Calculus. Both Isaac Newton in England, and Gottfried Leibniz in Germany, worked feverishly and mostly independently on the concept in the 17th century. They both produced manuscripts describing what we now call Calculus at about the same time, using different notations. Most contemporary observers consider both men to have developed a similar theory independently. Today, it is Leibniz’ \( \frac{dy}{dx} \) notation that is most widely used in mathematics, although Newton’s (and Euler’s) are also used.

\(^{14}\) The related, though not identical, terms supremum and infimum are also used in mathematics. The infimum is the greatest lower bound of a set of real numbers.
Criteria for Maximum

$\text{EQ A-21}$

For continuous and differentiable functions $y=f(x)$, the maximum and the minimum value of the variable $y$ will appear when the first derivative equals zero. (This is known as Fermat’s Theorem.)

Local extrema always have a tangent line with a slope of zero. The change in sign of the derivative (positive to negative for a maximum and negative to positive for a minimum) is what defines an extrema. Once the value of the extrema is discerned, one must check values to the right and to the left to see whether the derivative changes from positive to negative (increasing to decreasing for the original function) or vice-versa.

**Stationary Point Test**

This leads to two tests for a extremum, both based on the first-derivative $= 0$ conditions. For vector-valued functions, a stationary point is one at which all the first derivatives equal zero. For a single-variable function, there is a straightforward test based for a maximum based on the first derivative. (In both cases we are assuming a continuous and differentiable function.) These are used commonly enough to have an acronym: the first order condition (FOC) for an extremum.

Stationary Point

$\text{EQ A-22}$

$\mathbf{x}^0 = (x^0_1, x^0_2, x^0_3, \ldots)$ is a stationary point of $f(x)$ if

$f'_i(x^0) = 0, f''_i(x^0) = 0, f'_j(x^0) = 0, \ldots, f'_n(x^0) = 0; \quad \text{where:}$

$f'_i(x^0) = \frac{\delta f(x^0)}{\delta x^0_i}$
First Order Condition for Extremum (EQ A-23)

if \( f'(x) \geq 0 \) for \( x \leq c \), and
\[ f'(x) \geq 0 \] for \( x \geq c \), then
\[ x = c \] is a maximum point for \( f \)

---

c. The Economic Interpretation of the Derivative

In economics, the first derivative of a function is denoted the *marginal function*. The marginal function describes approximately by how much the function value changes when the independent variable changes by one unit. Investigation using marginal functions is called *marginal analysis*. *Comparative Statics* is a form of analysis that typically uses the derivatives of functions to analyze the behavior of individual variables in a system of equations.

---

**Section A.27. Calculus of Variations**

A classical approach to optimization using was developed by Euler and others in the 18th century. It involves a generalization of the differential calculus developed by Leibniz and Newton earlier. The fundamental problem in the calculus of variations is finding the maximum of a function that depends on two other variables, as well as the rate of change of one of these variables.

---

Calculus of Variations: Maximization Problem (EQ A-24)

\[
\max \int_{t=0}^{T} F(t, x, \dot{x})dt,
\]

where:
\[ x = x(t), \]
\[ \dot{x} = \frac{dx}{dt} \]}
A necessary condition for the solution to this problem is a function \( x = x(t) \) that satisfies of the *Euler equation*, (or *Euler-Lagrange equation*):\(^{15}\)

\[
\delta F = \frac{d}{dt} \left( \frac{\delta F}{\delta \dot{x}} \right) = 0
\]

The Euler equation often has a natural meaning in economic problems; see Section A.28., “Dynamic Programming.”

---

**Section A.28. Dynamic Programming**

**a. Introduction**

There are many problems that involve the aggregate effect of the outcome of a sequence of events. If a numerical value can be put on each individual outcome, the problem can be modeled as seeking the maximum (or minimum) of a discounted sum over time of the individual values of each event. Of course, interesting problems typically involve the ability of humans to affect the future outcomes by taking some action today.

To address these problems, the method of dynamic programming was developed by the brilliant American mathematician Richard Bellman [Bellman (1957)]. Bellman considered the task of an individual who seeks to maximize the benefits from some activity over time, when the policy choices made during one period affect the ability to enjoy benefits in the future. Although extremely difficult to implement in many settings (for reasons that are explained below), the analytical approach has now been applied throughout social and physical sciences.

The technical conditions for the use of the technique in economic applications was presented by Stokey & Lucas (1989); economic applications

---

are presented in Miranda & Fackler (2002) and Ljungqvist & Sargent (2000). The technique was adapted to the problem of business valuation by Anderson (2004b).

b. Sequence Problem and Functional Equation

The mechanics of multi-period problems of this type are generally represented by the following sequence problem:

\[
\sup_{\{x\}} \sum_{t=0}^{\infty} \beta^t f(s_t, x_t); \text{ [Sequence Problem]}
\]

where:

- \( t \) = time index;
- \( f(s_t, x_t) \) = reward function given state and action;
- \( s_{t+1} = g(s_t, x_t) \) transition function;
- \( \{s\} \) = sequence of state variables over time;
- \( \{x\} \) = sequence of control variables over time;
- \( \beta \) = a discount factor; \( 0 < \beta < 1 \)

Here the \( \sup \) operator represents the \textit{supremum}, or the highest value that can be obtained. We will assume that such a value exists within the set of real numbers and henceforth use the more common \( \max \) or \textit{maximum} operator.

\textit{Functional Equation}

The conditions under which the sequence problem can be recast as a series of individual optimization decisions has been studied extensively. Under specific conditions, the solution to the sequence problem in equation (EQ A-26) is the solution to the following \textit{functional equation}:
The equation (EQ A-27) is known as a functional equation because the expression $V(s,t)$ is not, strictly speaking, a function of just the variables $s$ and $t$, but instead the maximization of a family of functions. We will refer to it as the Bellman equation for this optimization problem.

c. Bellman equation with uncertainty

Stochastic dynamic programming is employed when the underlying transition or reward functions contain stochastic elements. The following is an example of a Bellman equation used in this manner:

<table>
<thead>
<tr>
<th>Functional Equation (EQ A-27)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V(s,t) = \max_x { f(s_t, x_t) + \beta E[V(s_{t+1}, t)] }$; [Functional Equation]</td>
</tr>
<tr>
<td>where:</td>
</tr>
<tr>
<td>$t = 0, \ldots, T$ is a time index;</td>
</tr>
<tr>
<td>$V(s, t) = \text{value at time } t \text{ given state } s$;</td>
</tr>
<tr>
<td>$f(s_t, x_t) = \text{a reward function given state and action}$;</td>
</tr>
<tr>
<td>$s_{t+1} = g(s_t, x_t)$ is a transition function;</td>
</tr>
<tr>
<td>$s_t = s_0, \ldots, s_T$ are state variables,</td>
</tr>
<tr>
<td>$x_t = x_0, \ldots, x_T$ are control variables; and</td>
</tr>
<tr>
<td>$\beta = \text{a discount factor; } 0 &lt; \beta &lt; 1$</td>
</tr>
</tbody>
</table>

The equation (EQ A-27) is known as a functional equation because the expression $V(s,t)$ is not, strictly speaking, a function of just the variables $s$ and $t$, but instead the maximization of a family of functions. We will refer to it as the Bellman equation for this optimization problem.

<table>
<thead>
<tr>
<th>Bellman Equation for Stochastic Dynamic Programming (EQ A-28)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V(s_t, x_t) = \max_x { f(s_t, x_t) + \beta E[V(s_{t+1}, x_{t+1})] }$;</td>
</tr>
<tr>
<td>where:</td>
</tr>
<tr>
<td>$\varepsilon = \text{random error term; and}$</td>
</tr>
<tr>
<td>$s_{t+1} = g(s_t, x_t, \varepsilon)$ is the transition function</td>
</tr>
</tbody>
</table>

16 This is best described in Rustagi (1994) and Chiang (1999). Chiang notes that this is not a mapping from real numbers to real numbers, which would be a function. Instead, it is a mapping from paths to real numbers, the real numbers being the quantities being optimized.
d. Euler equations

The intertemporal optimization decisions inherent in the functional equation are captured in the Euler equations for the functional equation:

\[ V'(s) = \frac{d[V(s,t)]}{dx} = 0 \]  

[first-order condition for maximum]

\[
0 = \frac{\delta f}{\delta x} + \beta \frac{d[V(s_{t+1})]}{dx} \\
= \frac{\delta f}{\delta x} + \beta \left( \frac{\delta V}{\delta s} \right) \left( \frac{dg}{dx} \right) \\
= f_s + \beta V_s g_s \]  

[Euler equation]

Here the first-order condition for a stationary point is that the derivative of the value function with respect to the control variable \( x \) vanishes. Differentiating the value function with respect to \( x \), using the chain rule of calculus on second term, we arrive at the Euler equation. This encapsulates the intuition that, at a maximum, the trade-off between enjoying immediate reward \( (f_s) \) is just offset by the discounted future benefits of postponing the immediate reward \( (\beta V_s g_s) \).

Additional Notes

- The Euler equation for this functional equation is an application of the broader Euler-Lagrange equation in the Calculus of Variations; see Section A.27., “Calculus of Variations.”
- Note that, if the functional equation incorporates a stochastic term, there is no direct application of the Euler-Lagrange equation and any “Euler equation” must be derived from a non-stochastic variant of the underlying equation.
- The marginal value of the state to the optimizer \( (V_s) \) is sometimes called the shadow price of the state.
- An additional condition called the envelope condition is often employed in analytical work.
- The solution to the functional equation typically requires an additional transversality condition which either forces the discounted value of
the reward function to zero at some point, or establishes a fixed termin-
ation reward and termination date.

e. Conditions for Solution of Functional Equation

Specific conditions for the solution of the functional equation have been identified in Stokey & Lucas (1989), and are not repeated here. Key elements of the business value functional that encourage (if not ensure) that the functional has a solution include:

1. A bounded reward function, which incorporates the natural limitations of human enterprises.
2. A gross discount rate that is between zero and one.
3. Limitations on the transition and reward functions that reflect the actual limitations in businesses.
4. The use of a parsimonious set of control and state variables.

SECTION A.29. Utility Functions

a. Variations in Functional Forms

Economists have used a handful of functional forms for analytical and empirical work on utility and risk aversion. These include exponential, power, quadratic, log, and linear (“affine”). Each specification implies certain characteristics, and the parameters used (whether supplied by assumption or by the data) further sharpen the meaning. 

However, there is no consensus on the functional form of the utility equation. This has very strong implications for empirical work, since estimating equations requires a specification, and a mis-specification of the equation can produce bogus results. This is one concern, for example, that arises out of the “equity premium puzzle” we discussed in Section 9.5., “Bubbles, Puzzles, and Troubles.”

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17 Lengwiler (2003, section 4.6) has a good summary of the various specifications and their consequences.

18 Worse, a mis-specified equation can produce seemingly good results (including the normal array of impressive-looking regression statistics) while actually proving very little about the relevant question.
We will return to this topic when we consider common models of investment behavior, such as the CAPM model of portfolio selection.

b. Mathematical Properties of Preferences

Preferences are usually assumed to display the mathematical properties listed below; for each property I give an informal description. Many microeconomic references provide rigorous definitions.\(^{19}\)

- Complete (Either the consumer prefers X, or prefers Y.)
- Reflexive (If the consumer could buy X, he will have a preference concerning it.)
- Transitive (If you prefer a big car over a small car, and a small car over no car, you will prefer a big car to no car at all.)
- Continuous (If you have a choice of one box of cookies and three boxes, you also have the choice of two boxes.)
- Monotonic or Weak Nonsatiation (“More is better;” if a consumer likes clothes, she will prefer more clothes to fewer clothes.)
- Convex (Consumers attribute diminishing marginal benefit to additional consumption of any one good.)

With a few qualifications, all these are either explicit, or implied by the assumptions listed in Section A.29., “Utility Functions.” The qualifications include observing how the monotonic (weak nonsatiation) assumption applies within a relevant range (known as “local nonsatiation”), and limiting the continuous and convex assumption to useful quantities.

c. Variations in Treatment of Risk\(^{20}\)

Utility functions differ on the attitude they presume about risk. These variations include:

- Variations in Absolute Risk Aversion
  We noted above the coefficient of absolute risk aversion. If there is no change in the risk aversion among consumers with different wealth, \(A'(w) = 0\) then we have constant Constant Absolute Risk Aversion (CARA). Alternatives are Increasing (IARA) and Decreasing Absolute Risk Aversion (DARA).

- Variations in Relative Risk Aversion

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19 See, e.g., Varian (1992).
20 This presentation is modeled after Lengwiler (2004, chapter 4).
A more subtle concept than variations in absolute risk aversion is variations in the relative aversion to risk, given risks that are the same fraction of wealth. For example, who would be more likely to wager 50% of his wealth on a lottery that cost 10%, a wealthy person or a poor one? Utility functions are sometimes characterized according to the implicit answer to this question as exhibiting Constant Relative Risk Aversion (CRRA), Increasing (IRRA) or Decreasing Relative Risk Aversion (DRRA).
Appendix B

Software for Business Valuation

by Patrick L. Anderson

An Introduction

This section deals with an important topic for both practitioners and researchers: the availability, usefulness, and power of software intended to assist them in their work. The following is a personal, yet quite extensive, set of notes on various types of software compiled by the author of The Economics of Business Valuation, Patrick L. Anderson.

SECTION B.1. Admonition: “No software is a substitute for thinking”

Businesses had value long before computers were invented; investors and entrepreneurs organized, bought, sold, and managed businesses for centuries without the aid of any software. While computer software, hardware, and networks wonderfully increase the ability to acquire and manipulate data, they cannot substitute for thinking.

There are many valuation, accounting, and finance texts (including coursework for credential-giving organizations) that attempt to make it easier for valuation practitioners by pre-digesting certain techniques into spreadsheet models or handy formulas. Much of this is truly useful information. However, combining these formulas with software that automates the creation of a valuation report is an open invitation to stop thinking.

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Unfortunately, overreliance on computer software, including pre-set spreadsheet models and pre-packaged “valuation” software, has become a serious problem in valuation and finance. The use of these as “black box” models violates numerous standards of ethical behavior, information quality, and civil procedure.  

Therefore, I must admonish valuation practitioners that no software is a substitute for thinking. Of course, if you reached this companion volume, you have read a good portion of the EBV book, and probably already agree with this!

**SECTION B.2. Special Tools Used in the The Economics of Business Valuation**

Almost all of the value functional examples in *The Economics of Business Valuation* were prepared using custom-programmed routines within the MATLAB® computer software environment.

These routines were sometimes used in conjunction with the following toolboxes (collections of routines) prepared by different parties for use within MATLAB®: Rapid Recursive®, CompEcon, MDP, and Business Economics toolboxes. These are all described further below. In a few cases, the author modified or substituted certain custom-programmed functions for those that were part of the third-party toolboxes.

**d. Specialized Software Tools for Novel Techniques**

The specialized tools used in EBV include the following:

1. **MATLAB®, a vector-processing mathematical software**
   Many of the examples in this book were done using MATLAB®,

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21 A number of US government agencies have formal information quality standards, which require analyses to be supported by data and methodology. The U.S. Information Quality Act (Section 515 of Public Law 106-554, 2000) requires the adoption and use of such standards in certain activities involving expert opinion or technical reports. In addition, the (U.S.) Federal Rules of Civil Procedure require (in Rules 26 and 702) that expert testimony include the data and methodology to support any conclusion. Most states have similar rules. Both state and federal courts, in addition, have developing case law on the standards for expert testimony.
including projections, quadrature, optimization, Monte Carlo, financial options calculations, as well as graphical illustrations and symbolic calculus. Within this software environment, I used routines from the Rapid Recursive®, Business Economics, CompEcon, and MDP toolboxes discussed below.

MATLAB® is available from the Mathworks, whose web site is:
http://www.mathworks.com

MATLAB® and similar software are described below under Section B.4. "Notes on Mathematical Software."

2. The Rapid Recursive® Toolbox for MATLAB®, a sequential decision problem composition and solution software

This is the main software tool used in the composition and solving the value functional problems included in The Economics of Business Valuation, and also the additional examples included in this Solutions Manual. One motivation for its development was to perform exactly these tasks. Prototypes of the toolbox were used in the examples in The Economics of Business Valuation, while a commercial version was available for most of the examples in the Solutions Manual.

The Rapid Recursive® Toolbox is available from Supported Intelligence, LLC, whose web site is:
http://www.supportedintelligence.com

This software is further described in “Rapid Recursive® Toolbox for MATLAB®” on page 4.

3. Computational Economics (CompEcon) Toolbox

This set of MATLAB® routines for function approximation, dynamic programming, root finding and optimization was created for the book Applied Computational Economics and Finance (MIT Press, 2002) by Mario J. Miranda & Paul L. Fackler. The toolbox is available at:
http://www4.ncsu.edu/~pfackler/compecon/toolbox.html

4. Markov Decision Processes Toolbox

This set of MATLAB® routines was prepared by Iadine Chadès, Marie-Josée Cros, Frédérick Garcia, and Régis Sabbadin at the Institut National de la Recherche Agronomique (INRA) in Paris, France. The toolbox is available at:
5. Business Economics Toolbox

The author created a toolbox for use with MATLAB® that implements many of the routines discussed in the book *Business Economics & Finance* (2004). Some of these routines were used in the examples and other analyses in *The Economics of Business Valuation*, and it would be useful to readers of that book as well. The Business Economics Toolbox has been available since the 2005 under a no-additional-cost license to purchasers of *Business Economics & Finance*, from the following website:

http://www.andersoneconomicgroup.com

The author intends to update the aforementioned toolbox, and may offer it in the future to purchasers of *The Economics of Business Valuation*.

Please note that each of these toolboxes represent many hours (if not years) of work by their respective authors and contributors, and the license terms under which they are offered restrict some uses. The authors or distributors may charge a license fee to users, or may do so in the future. Please respect the intellectual property of the authors of these toolboxes by making a good faith effort to follow the terms of the license agreements under which they are offered.

e. Standard Tools

The examples presented in this book were developed using both standard tools and novel applications of tools that have not become standard among practitioners. I consider the following spreadsheet software to be a standard tool for practitioners, and it was used in the manner discussed under Section B.5. ”Spreadsheets” on page 8.

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**SECTION B.3. Rapid Recursive® Toolbox for MATLAB®**

The Rapid Recursive® Toolbox for MATLAB® was used in prototype form to compose, check, and solve most of the value functional examples given in the text of *The Economics of Business Valuation*. In the commercially-released form, it was used to do the same for most of the examples provided in this *Solutions Manual*. 
The toolbox, which runs on the MATLAB® platform, allows users to compose, check and solve sequential decision problems, and report the results in a clear manner. It is the first commercial software of its type, and should allow a dramatic expansion of the number of people able to actually use value functional or “recursive” methods.

The Rapid Recursive® Toolbox is available from Supported Intelligence, LLC, a company founded by the author with the intention of bringing this kind of innovative software to the general market. The web site for the firm is:

http://www.supportedintelligence.com

The Rapid Recursive® software assists the user in undertaking the most difficult and time-consuming tasks in applying the theory of the value functional approach: composing the problem, error-checking it, and formulating it into a mathematical equation that can be solved. The software also solves the problem (after checking to ensure that a solution is possible), using one of at least two possible solution algorithms (value functional iteration and policy improvement). Finally, it provides for a standardized report of the solution of the value functional equation, consisting of a table of possible states, the value in each state, and the value-maximizing policy.

A number of Solution Templates are bundled with the software in an open-code format. These compose, solve, and then report the results of such problems as the valuation of a start-up company, when to replace expensive machinery or computer hardware, when to shut down a mine or similar declining asset, and the value-maximizing policy for the owner of rental property receiving multiple rent offers each month.

These tools have been tailored for use in investment and business valuation decisions, but are general enough to be used in a broad range of applications in engineering, computer science, control systems, agriculture, and biology, as well as for medical decision making and analysis of economic policy.

Like the other toolboxes mentioned above, this one requires the use of MATLAB®, and therefore will largely be used by quantitatively-skilled users. However, it points the way toward a much broader use of this method in the future.
SECTION B.4. Notes on Mathematical Software

a. Vector-processing Mathematical Software

Uses
Vector-processing mathematics programs natively handle complex mathematical tasks, including linear algebra, function definition and execution, and specialized calculations. They often can perform optimization routines involving nonlinear relations and constraints. Although some option valuation calculations can be done with a spreadsheet, it is generally better to do them in a mathematical software environment.

Software
We have used MATLAB®, a product of the Mathworks Company, for the mathematical routines such as option valuation and dynamic programming described in this book. There are a handful of very powerful special routines already programmed into MATLAB® toolboxes. These are available from the Mathworks or other authors.

Alternatives to MATLAB® include: Mathematica, Octave, Gauss, and a handful of other vector-processing software environments.

Limitations
The primary limitations of this type of software are: high cost (for commercial packages), extensive training requirement, difficulty in use, and often obscure command syntax.

b. Simulation Software

Uses
Mathematical software of the type described above can usually perform true simulation, including the generation of random numbers that cause other calculations in an extensive model. However, certain programs are designed specifically for simulation, and can be used directly with financial and business concepts.
Software; Limitations

These include Simulink (another program of the Mathworks) which this author has used in applications like those described here. It has similar limitations as the MATLAB® software discussed above. Many of the programs listed in Section B.6. “Monte Carlo & Decision Tree Software” on page 9, also perform simulations.

I have not included here many simulation programs designed primarily for engineering applications.

c. Statistics and Econometrics Software

Uses

Statistics programs natively handle time-series and cross-section data that would prove unwieldy, or impossible, to handle in a spreadsheet. They are designed to perform statistical calculations (including statistical calculations done primarily on time-series economic data, which are often called “econometric” calculations, even if they are identical to those done for non-economic applications.) In general, they do these tasks much better than programs designed for other purposes.

Software

There are numerous statistical software packages available, including Stata, Eviews, SAS, RATS, and others. No specific statistical software is necessary for the problems described in this book. In addition, most mathematical programs (such as MATLAB® and Mathematica) have specific statistical and econometric functions, and those familiar with them may prefer to perform such analyses in that environment. Some also perform simulations (see discussion above).

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22 Interested readers should get a copy of Business Economics and Finance: with MATLAB®, GIS and Simulation Models (CRC Press, 2004), which contains a number of chapters related to techniques of business valuation, and outlines specific simulation and mathematical models that can be used for this purpose. Of course, (given the admonition above), the underlying theory is explained first.
Limitations

The primary limitations of this type of software are: moderately high cost when compared with spreadsheet programs (for commercial packages), training requirements, and relative usage difficulty.

SECTION B.5. Spreadsheets

Uses

Spreadsheets are wonderful tools for calculating data (in particular accounting data), that can neatly be organized into rows and columns. A spreadsheet is often the best tool for tasks largely involving tabular data, and those which require simple arithmetic operations. It is also often the best tool to report such data, after simple analyses and calculations.

Software

In addition to the ubiquitous Microsoft Excel®, there are a number of other spreadsheet programs, including the OpenOffice suite of programs and the Numbers program distributed by Apple Computer. These are nearly all highly usable and reliable when used as intended, and the selection of one is largely a personal preference.

Limitations

Although it is possible to push spreadsheet software into performing calculations beyond simple math, we recommend against it when reliability is important. It is better to use other software to execute statistics, simulation modeling, forecasting, option valuation, or other such routines that involve nonlinear functions, optimization, or complex calculations that cannot be shown within the cells of a worksheet.23 It is also better to use other software when very large data sets must be imported. Finally, when very large numbers (or large sets of numbers) are used, be aware

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23 This advice does not apply to routines that are called by a spreadsheet, and where the analyst has the capability of reviewing the calculations in the other program. A number of mathematical, statistical, DTA, and similar programs have extensions that allow this interaction, and nearly all read and write to spreadsheet file formats.
that errors and data limitation have been discovered even in the best-known spreadsheet software, Microsoft Excel.\textsuperscript{24}

**Freely-Available Templates and the Black Box Problem**

There are many calculation templates that purport to perform valuation calculations available from various websites. Some of these websites contain material from bona fide experts in the field. However, one should be wary of spreadsheet templates that effectively become a “black box” model. In the hands of a person who knows what he or she is doing, and who can explain every step of the calculation, some of these templates are fine. In other hands, they are not—even if the original template was used by a well-known expert.

If you do not know the structure of the underlying calculations, \textit{you should not rely upon the template}. That advice is doubly apt for a template prepared by another party.

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**SECTION B.6. Monte Carlo & Decision Tree Software**

**Uses**

\textit{The Economics of Business Valuation} demonstrates the failure of the standard discounted cash flow method to properly value an investment opportunity when real options are present. Alternatives to DCF methods that have been applied to real option problems include two methods that are described here. The recursive approach, which typically requires software of the type described above.

Event trees are difficult to model using standard spreadsheet software. While many programs exist to \textit{illustrate} an event tree, relatively few natively incorporate the features necessary to use Decision Tree Analysis (DTA) or assess real options in practical situations that contain multiple decision nodes. Monte Carlo techniques can be used to increase the

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\textsuperscript{24} For example, the author was once confronted by a claim that he had analyzed over 64,000 different data points supplied in a spreadsheet—but that the original data set was larger, and had been inadvertently truncated by the spreadsheet software. Here the software’s limitations were unknown to the original compiler of the data, and therefore to everyone to whom the data were given afterwards.
number of potential “leaves” on an event tree, and include more interactions than are typically possible on a single decision tree.

Software designed for DTA and Monte Carlo analysis of scenarios can significantly improve the speed and capability of a practitioner. The better packages provide support for common situations, allow for multiple approaches and optimization algorithms, and provide transparency to the user on key calculations and assumptions.

**Software**

Software of this type include: DPL (from Syncopation Software); Risk Simulator (from Real Options Valuation); @RISK and PrecisionTree (from Palisade). Other simulation software with similar capabilities include: GoldSim (from GoldSim Technology Group); Simulink (from Mathworks), also discussed in the section on Mathematical Software; and Lumenaut (from Lumenaut, Ltd.).

In addition, there are a good number of low-cost and shareware templates for simple real and financial options that are available.

**Limitations**

These programs vary widely in capability, ease of use, transparency, and algorithms. Some function as add-ons to a spreadsheet program, while others have their own algorithms. Users should be particularly sensitive to the transparency of the calculations and the admonition above.

In addition, I have not included here the many financial option pricing programs. If you trade financial options regularly, you should probably invest in learning and using a program dedicated for this purpose.

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**SECTION B.7. Advice: “Business Valuation” Software**

**Uses**

Software of this type is advertised as a means by which to mechanize some of the tasks that are often necessary to perform a business valuation, such as assembling data on comparable companies, creating schedules,
collecting information on the industry and economy, and performing calculations using specific valuation models. Many programs will also create a report incorporating the inputs and calculations into a pre-written template. The report templates may include statements asserting that the analysis was done in accordance with specific professional standards.

Marketers of some of these programs claim they can do nearly all the work required for a business valuation, frequently promising that entire reports can be done “in minutes” after entering only a small amount of data. Indeed, the time-saving and report-writing features of the software, not the validity of any resulting opinion, appear to be the major selling points. The website for one claims that the product is intended for those “who want to determine the fair market value of a business but don’t want to get bogged down in valuation theory and complicated mathematics.”

**Limitations; the “black box”**

A software program or other analytical method, which transforms input data into an output without revealing the inner workings, is known as a “black box.” In general, social science practitioners should not rely on such devices, because there is no way to know how the results were obtained. It would appear from the marketing claims (excerpted in the footnotes) that users of many of these products can generate a valuation report in a matter of minutes, from a minimal amount of data, and with

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25 Here are some statements about four different business valuation software products:

- “Estimate your business value in just a few moments”; “Calculate your business valuation results instantly by entering a few numbers” [This is a description of a product that requires, apparently, only two sets of inputs from the practitioner]
- “We have performed a valuation engagement, as defined in the Statement on Standards for Valuation Services (SSVS) of the American Institute of Certified Public Accountants...” [This text from a report template generated by another program]
- “[Product] can generate a 50+ page draft report in minutes, based on the specific fact pattern of your engagement.” [Report is also claimed to fulfill requirements of USPAP].
- “If you can enter numbers into a table, you can successfully determine the valuation of a business.” [Product purports to produce valuations that comply with IRS RR 59-60.]

All statements retrieved from product websites in November 2010, located by performing a search for “business valuation software.” Many more similar statements exist for these and similar products.
little or no actual analysis. This is a “black box” of which I was critical above.

The content and cost of these programs vary widely. For example, some provide useful comparison data for some companies in some industries; some include plentiful suggestions in their report templates; some require more work from a practitioner than others before they will generate a report. Therefore, at least some of them can be used properly by an ethical and skilled practitioner. However, they are almost certainly used as a “black box” by other practitioners. Furthermore, the pre-written claims in the report templates about the practitioner following specific “standards” are obviously false when the program is used as such.

I am hopeful that these products will improve significantly in the future, and that ethical practitioners will refuse to purchase products that promise to do their thinking for them.

SECTION B.8. Other Useful Software Not Considered

Here

There are other software products that fit in categories that overlap with the software discussed here. As the focus in this volume is on software used in business and investment valuation, these other categories of software have not been considered, even though they may be essential for the management, accounting, risk quantification, and regulatory compliance purposes.

In particular, we did not consider the following categories of software in this chapter:

- Accounting
- Project management
- Financial option valuation
- Mortgage instrument, stock or bond portfolio analysis
- Enterprise Resource Planning
- Software that is intended to collect and display data (such as in “dashboards”), or to help organize teams or funnel information to the right
person in an organization, which is sometimes called decision support software.

- Regulatory compliance, especially compliance with regulations for financial institutions (such as for capital adequacy, sufficient collateralization or margin, or meeting specific tests set in regulation or law)
- Industry-specific templates for use in most of the software considered in this chapter
- Software that primarily reports industry or market data on prices
- Asset-liability management

Some programs within these categories perform functions that are covered by more purpose-specific programs discussed in other categories.

Section B.9. General Disclaimers; Trademarks

a. General Disclaimers on Software Appendix

In this appendix are listed some applications software, available at the time of writing, which could be useful to the valuation practitioner. This information is provided with the following, somewhat obvious, understandings:

- This information is based on the experience of the contributors to this volume, as of the time of writing. As noted further below, neither these individuals, nor the publisher, recommend the use of these products except by qualified individuals in settings appropriate for their use. In many cases, that will not include settings contemplated by readers.

- The features and availability on certain operating systems, as well as other aspects of these programs will change over time.

- We have not conducted an exhaustive search of all available products. Other products promising similar features will exist in the marketplace in the future, and there may be others in the marketplace at the current time. I urge readers to consider these as well.

- Neither the author nor the publisher offer support for any of these products, nor any warranty that a reader will be able to use tech-
niques described in this book, with or without the software described here, successfully in any specific setting.

• The statements about the uses and limitations of certain categories of software, and the use of certain categories of software, are opinions of the author of this chapter. Readers are cautioned that this is an inherently subjective topic, and are invited to perform their own search and make their own decision about software usage.

b. Specific Disclaimer for Novel Software

A number of the novel techniques described in this book have been developed recently. Some have been developed by the author of *The Economics of Business Valuation*.

Below, we list a newly-available commercial software product that allows for the composition, error-checking, and solution of business valuation problems using the “value functional” or “recursive” method. This software was developed initially to solve exactly these types of problems, by the author.

However, this software is still quite new. Readers attempting to replicate the results of calculations demonstrated in this book and *The Economics of Business Valuation* will likely encounter difficulties that are common to those using new techniques and novel, experimental, or specialized software.

c. Trademarks

Where possible and known to the author, we have included trademark symbols for the names of products with registered trademarks in the United States. Other names are trademarks of their respective providers.

d. Future Revisions; Other Products; Corrections

Although this is an inherently subjective topic, and we have made a good-faith effort to identify selected software products and their uses and limitations properly, it is almost certainly the case that we have missed some products or mis-identified others. Therefore, we offer the following means to assist us in improving this appendix in a future revision:

• Providers of software products that are not listed here, but fall into one of the categories listed, may contact the editor using the contact information listed later in this volume, and request that it be included in a
future revision. Please supply the editor with information on the product, platform, features, and web site. Note that, as we do not intend this appendix to be an exhaustive list, products listed must either have a large user base, unique features, or other particular benefit to command inclusion.

Note that a website promising a product alone will not constitute sufficient evidence that a product exists and works as advertised. Please do not submit information on beta releases, planned software, or periodic revisions of software already mentioned here. We reserve the right to determine which products to list, and which not to list, based on our own judgement.

- Suppliers of products listed here whose names are registered trademarks, where the text does not include the registration mark, can inform the editor of that registration using the contact information later in this volume.

- If you are the provider of a software product and believe your product has been described inaccurately, please contact the editor in writing and identify the specific error in the text, the basis for this claim, and a suggested correction to the text. We pledge to correct any confirmed errors brought to our attention in this manner in the next revision.

Note that claims about product features that are not mentioned here, differences of opinion about what is the best use for a category of software, or assertions about products that cannot be easily confirmed, do not constitute an “inaccuracy.”

Although we plan to release a revision of this volume in the future, the editor and publisher do not promise to release revisions on any particular schedule.
Appendix C

Errata

Introduction

Any author will inevitably tread over ground that is considered sacred to someone else. This is the case with The Economics of Business Valuation (EBV), as well as this companion volume. The EBV author, as well as all others included in the Solutions Manual, have attempted to properly present material accurately.

Inevitably, there will be some qualifications and corrections that, in a perfect world, would have been anticipated. Furthermore, there will be differences in how certain techniques are supposed to be used, and how they are actually used, which is bound to produce some ambiguities as to whether the theory, or the practitioner, is correct. However, this chapter attempts to address corrections and clarifications for both EBV and this companion volume.

Section C.1. Corrections or Additions to The Economics of Business Valuation

In this section we list errors and clarifications for The Economics of Business Valuation text.

Appendix B “Guide to the Solutions Manual” (p. 337)

Appendix B mentions an extended discussion of antecedents of the New Present Value (NPV) rule, which was actually included in the EBV text itself, rather than placed in the companion volume. For the reader interested in this discussion, see EBV pages 39-41.

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Appendix B “Guide to the Solutions Manual” (p. 337)

Appendix B mentions a history of option contracts, which was actually included in the EBV text itself, rather than placed in the companion volume. For the reader interested in this discussion, see EBV pages 170-172.

Appendix C “Description of Subject Companies” (p. 345)

In Table C.5 “Exxon Mobil income statement, 2006-2010,” the net income per common share for 2010 is a typo. It reads $624.00, when it should be $6.24.

SECTION C.2. Suggestions for Book Authors and Errors

In addition to covering a broad range of topics, EBV laid out the argument for a new theory, as well as explained a new technique. Such novelty necessarily brings with it a difficulty in explanation. 26 If you have any suggestions for the book or companion volume authors, or find any errors, please contact us via email: egrover@andersoneconomicgroup.com.

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26 For this reason we tried to provide additional examples and explanations in this companion volume. See Chapter 6 “Applications.”